On a model for the storage of files on a hardware II : Evolution of a typical data block.

Vincent Bansaye

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Abstract

We consider the generalized version in continuous time of the parking problem of Knuth introduced in [1]. Files arrive following a Poisson point process and are stored on a hardware identified with the real line, at the right of their arrival point. We study here the evolution of the extremities of the data block straddling 0, which is empty at time 0 and is equal to $\mathbb{R}$ at a deterministic time.

Key words. Parking problem. Data storage. Random covering. Poisson point process. Lévy process.

A.M.S. Classification. 60D05, 60G55, 60J80, 68B15.

1 Introduction

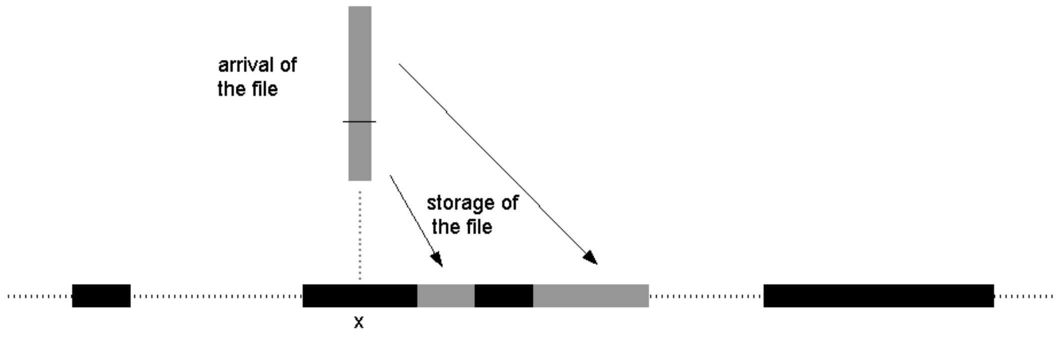
This paper is a continuation of [1] but it can be read independently. We consider a generalized version in continuous time of the original parking problem of Knuth, as a model for the storage of files on a hardware. We are interested in the evolution of a typical data block while files are stored on the hardware and we shall characterize the process of the extremities and the length of this block.

We recall now the process of storage of files. In the original problem of Knuth, files arrive successively at location chosen uniformly among n spots. They are stored in the first free spot at the right of their arrival point (see [6, 8, 9]). In the model considered here, the hardware is identified with the real line and a file labelled i of length (or size) l_i arrives at time t_i on the real line at location x_i. The storage of this file uses the free portion of size l_i of the real line at the right of x_i as close to x_i as possible (see Figure

*Laboratoire de Probabilités et Modèles Aléatoires. Université Pierre et Marie Curie et C.N.R.S. UMR 7599. 175, rue du Chevaleret, 75 013 Paris, France.
e-mail : bansaye@ccr.jussieu.fr

1). That is : it covers $[x_i, x_i + l_i[$ if this interval is free at time t_i. Otherwise it is shifted to the right until a free space is found and it may be split into several parts which are stored in the closest free spots.

Figure 1. Arrival and storage of a file on the hardware, where the data blocks are represented by black rectangles.



The arrival of files follow a Poisson point process (PPP) : $\{(t_i, x_i, l_i) \ : \ i \in \mathbb{N}\}$ is a PPP with intensity $\mathrm{d}t \otimes \mathrm{d}x \otimes \nu(\mathrm{d}l)$ on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$. We denote $\bar{\nu}(x) = \nu(]x, \infty])$ and we assume $\mathrm{m} := \int_0^\infty l\nu(\mathrm{d}l) < \infty$. So m is the mean of the total sizes of files which arrive during a unit interval time on some interval with unit length. In [1], this random covering has been constructed rigorously and some statistics of this covering were given. We proved that the hardware becomes full at a deterministic time equal to $1/\mathrm{m}$, studied the asymptotics at this saturation time and characterized the distribution of the covering at a fixed time by giving the joint distribution of the block of data straddling 0 and the free spaces on the sides of this block.

In this work, we focus on the dynamics of the covering and we shall study the block of data straddling a typical point, say 0 for simplicity, which is denoted by $\mathbf{B}_0$. Thus $\mathbf{B}_0(t)$ is the block of data of the hardware containing 0 at time t. We will show that its extremities and its length are pure jump Markov processes.
Specifically, if a file arrives at time t at the left of $\mathbf{B}_0(t-)$ and cannot be stored entirely at its left, it yields a jump of the left extremity of $\mathbf{B}_0$. The data of this file which cannot be stored at the left of $\mathbf{B}_0(t-)$ are called *remaining data*. These remaining data yield a jump of the right extremity of $\mathbf{B}_0$ (see Figure 2). We shall prove that these events happen at instants which accumulate at $1/\mathrm{m}$ and induce a random partition of the time interval $[0, 1/\mathrm{m}]$ with the Poisson-Dirichlet distribution (Theorem 2) and that the jumps of the extremities at these instants form a PPP on $[0, 1/\mathrm{m}] \times \mathbb{R}_+ \times \mathbb{R}_+$ (Proposition 2). Moreover the successive quantities of remaining data form an iid sequence (Corollary 2).
If a file arrives on $\mathbf{B}_0$, it yields a jump of the right extremity only (see Figure 3). The other files do not induce immediately a jump of $\mathbf{B}_0$ and we get the evolution of $(\mathbf{B}_0(t))_{t\geq 0}$ (Theorem 4). Finally, we prove that the process describing the length of $(\mathbf{B}_0(t))_{t\geq 0}$ is a branching process with immigration (Corollary 5).

Figure 2. Jumps of the extremities of $\mathbf{B}_0$ ($\Delta g(t)$ and $\Delta d(t)$) and remaining data induced by the arrival of a file at time t at the left of $\mathbf{B}_0(t-)$.

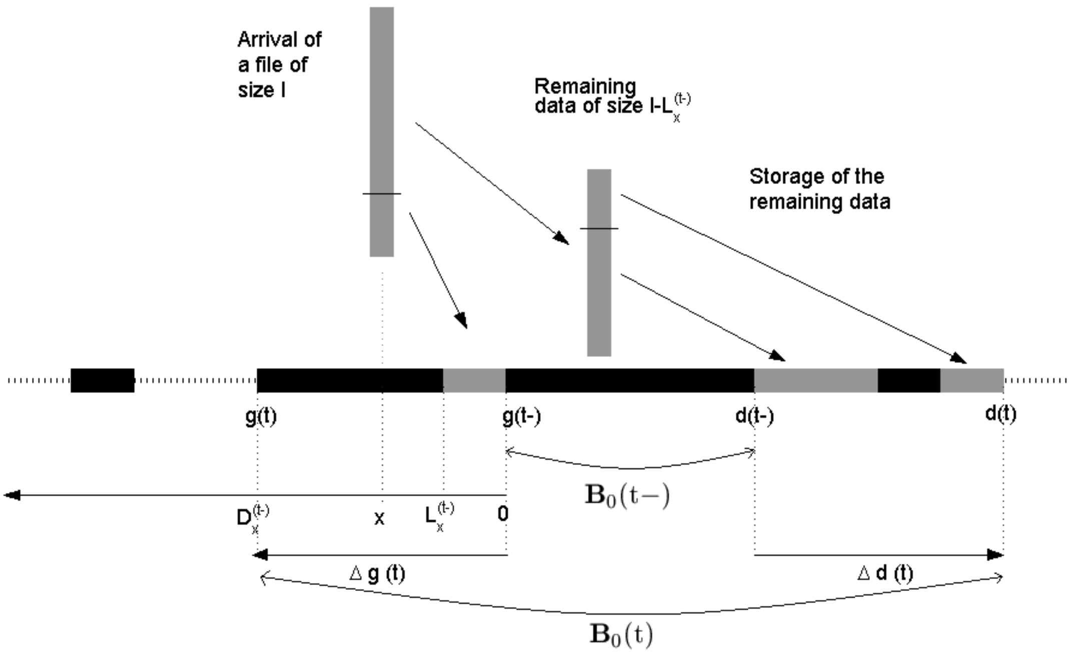
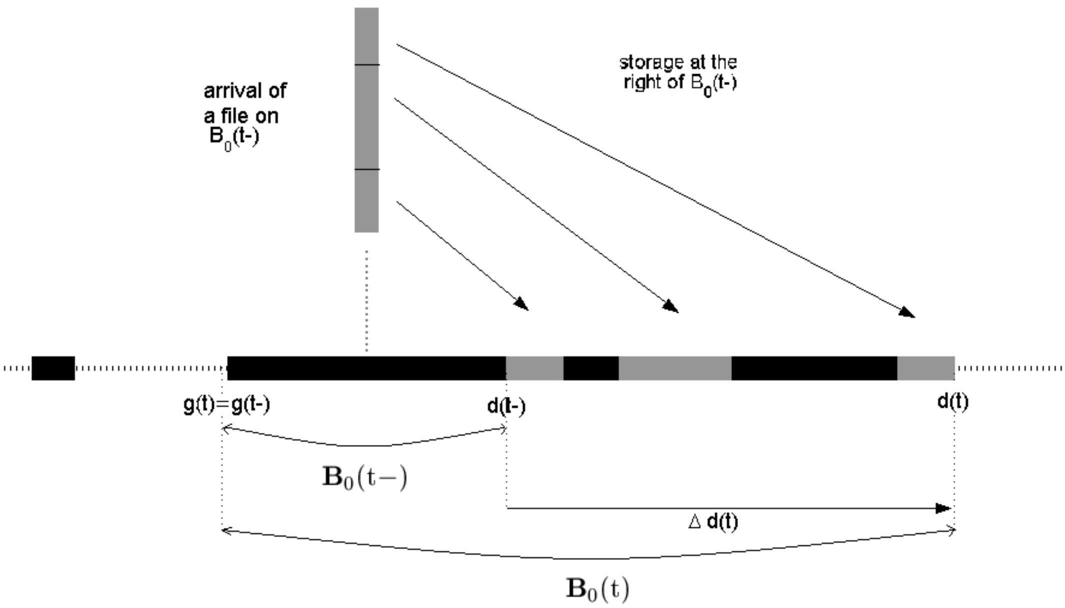


Figure 3. Jump of the right extremity of $\mathbf{B}_0$ ($\Delta d(t)$) induced by the arrival of a file at time t on $\mathbf{B}_0(t-)$.



2 Preliminaries

The covering $\mathcal{C}(t)$ described in Introduction has been constructed in Section 2.1 in [1] and we recall some useful results of this work. We denote by $\mathcal{R}(t)$ the complementary set of $\mathcal{C}(t)$. It is natural and convenient to decide that files and so $\mathcal{C}(t)$ and $\mathcal{R}(t)$ are closed at the left, open at the right. We introduce the process $(Y_x^{(t)})_{x\in\mathbb{R}}$ defined by

$$Y_0^{(t)} := 0 \quad ; \quad Y_b^{(t)} - Y_a^{(t)} = \sum_{\substack{t_i \le t \\ x_i \in]a,b]}} l_i \quad - \quad (b-a) \quad \text{for } a < b. \tag{1}$$

It has càdlàg paths and stationary independent increments. The process $(Y_x^{(t)})_{x\geq 0}$ is then a Lévy process. Its drift is equal to -1 and its Lévy measure is equal to $t\nu$. Its Laplace exponent $\Psi^{(t)}$ defined by

$$\forall\ \rho \geq 0, \quad \mathbb{E}(\exp(-\rho Y_x^{(t)})) = \exp(-x\Psi^{(t)}(\rho)), \tag{2}$$

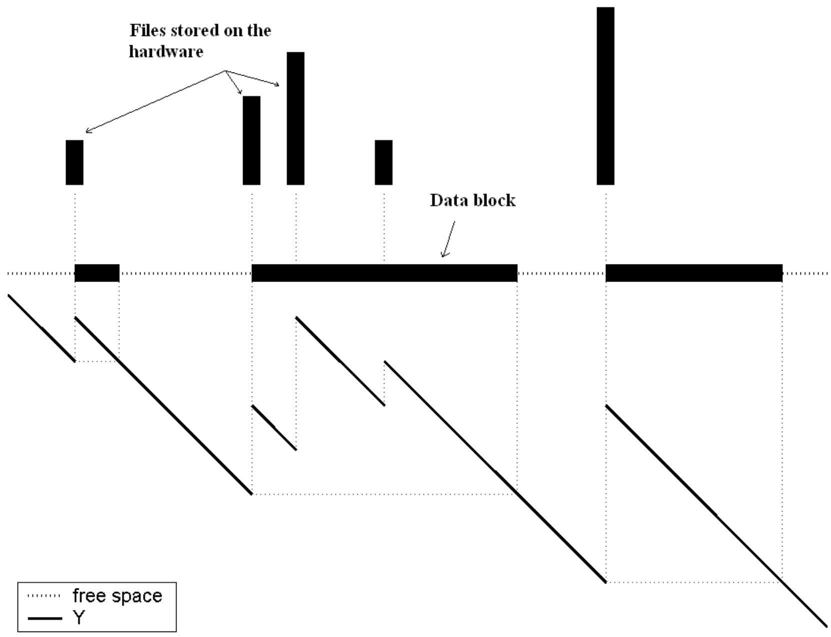
is given by

$$\forall\ \rho \geq 0, \quad \Psi^{(t)}(\rho) = -\rho + \int_0^\infty \left(1 - e^{-\rho x}\right) t\nu(\mathrm{d}x). \tag{3}$$

Introducing also its infimum process $I_x^{(t)} := \inf\{Y_y^{(t)} : y \leq x\}$ for every $x \in \mathbb{R}$, we got the following expression for the covering and the free space

$$\mathcal{C}(t) = \{x \in \mathbb{R} : \ Y_x^{(t)} > I_x^{(t)}\}, \qquad \mathcal{R}(t) = \{x \in \mathbb{R} : Y_x^{(t)} = I_x^{(t)}\} \quad \text{a.s.} \tag{4}$$

Figure 4. Representation of Y on a part of the hardware.



The time when the hardware becomes full is equal to 1/m, that is a.s $\mathcal{C}(t) = \mathbb{R}$ iff $t \geq 1/\mathrm{m}$. Thus we already know that $\mathbf{B}_0(0) = \varnothing$ and $\mathbf{B}_0(1/\mathrm{m}) = \mathbb{R}$ and we shall study $(\mathbf{B}_0(t))_{t\in[0,1/\mathrm{m}]}$. In that view, we introduce $g(t)$ (resp. $d(t)$, resp. $l(t)$) the left extremity (resp. the right extremity, resp. the length) of the data block containing 0 :

$$\mathbf{B}_0(t) = [g(t), d(t)[, \qquad l(t) = d(t) - g(t).$$

We will also need the free space at the right of $\mathbf{B}_0(t)$ denoted by $\overrightarrow{\mathcal{R}(t)}$ and at the left of $\mathbf{B}_0(t)$, turned over, closed at the left and open at the right, denoted by $\overleftarrow{\mathcal{R}(t)}$. If $\mathcal{R} \subset \mathbb{R}$ and $\mathcal{R} = \sqcup_{n\in\mathbb{N}}[a_n, b_n[$, we denote by $\widetilde{\mathcal{R}} = \sqcup_{n\in\mathbb{N}}[-b_n, -a_n[$ the symmetric set closed at the left and open at the right. Then we can define (see Section 3 in [1] for details)

$$\overrightarrow{\mathcal{R}(t)} := (\mathcal{R}(t) - d(t)) \cap [0, \infty], \qquad \overleftarrow{\mathcal{R}(t)} := \widetilde{\overrightarrow{\mathcal{R}(t)}},$$

which satisfy the following identity

$$\mathcal{R}(t) = (d(t) + \overrightarrow{\mathcal{R}(t)}) \sqcup (\widetilde{-g(t) + \overleftarrow{\mathcal{R}(t)}}). \tag{5}$$

In [1] Section 3, we proved that $\overrightarrow{\mathcal{R}(t)}$ and $\overleftarrow{\mathcal{R}(t)}$ are the range of the processes $(\overrightarrow{\tau}^{(t)}_x)_{x\geq 0}$ and $(\overleftarrow{\tau}^{(t)}_x)_{x\geq 0}$ respectively defined by

$$\overrightarrow{\tau}^{(t)}_x := \inf\{y \geq 0 : |\overrightarrow{\mathcal{R}(t)} \cap [0, y]| > x\}, \qquad \overleftarrow{\tau}^{(t)}_x := \inf\{y \geq 0 : |\overleftarrow{\mathcal{R}(t)} \cap [0, y]| > x\}.$$

Moreover denoting by $\kappa^{(t)}$ the inverse function of $-\Psi^{(t)}$ and by $\Pi^{(t)}$ its Lévy measure :

$$\kappa^{(t)} \circ (-\Psi^{(t)}) = \mathrm{Id}, \qquad \forall \rho \geq 0, \quad \kappa^{(t)}(\rho) = \rho + \int_0^\infty (1 - \mathrm{e}^{-\rho \mathrm{x}})\Pi^{(t)}(\mathrm{dx}), \tag{6}$$

enabled us to describe $\mathcal{R}(t)$ in the following way :

Theorem 1. *(i) The processes $\overrightarrow{\tau}^{(t)}$ and $\overleftarrow{\tau}^{(t)}$ are two indepedent subordinators with Laplace exponent $\kappa^{(t)}$, which are independent of $(g(t), d(t))$.*
(ii) The distribution of $(g(t), d(t))$ is specified by :

$$(g(t), d(t)) = (-Ul(t), (1 - U)l(t)),$$

$$\mathbb{P}(l(t) \in \mathrm{d}x) = (1 - \mathrm{m}t)\big(\delta_0(\mathrm{d}x) + \mathbb{1}_{\{x>0\}} x \Pi^{(t)}(\mathrm{d}x)\big)$$

where U uniform random variable on $[0, 1]$ independent of $l(t)$.

For the basic example $\nu = \delta_1$, we got for all $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$\mathbb{P}(Y^{(t)}_x + x = n) = e^{-tx} \frac{(tx)^n}{n!}, \tag{7}$$

$$\mathbb{P}(\overrightarrow{\tau}^{(t)}_x = x + n) = \frac{x}{x + n} e^{-t(x+n)} \frac{(t(n + x))^n}{n!}, \qquad \Pi^{(t)}(n) = \frac{(tn)^n}{n.n!} e^{-tn} \tag{8}$$

Thus $l(t)$ follows a size biased Borel law :

$$\mathbb{P}(l(t) = n) = (1 - t) \frac{(tn)^n}{n!} e^{-tn}.$$

We proved also the following identities :

$$\bar{\Pi}^{(t)}(0) = t\bar{\nu}(0), \qquad \int_0^\infty x\Pi^{(t)}(\mathrm{d}x) = \frac{\mathrm{m}t}{1-\mathrm{m}t}, \qquad [\kappa^{(t)}]'(0) = \frac{1}{1-\mathrm{m}t}, \tag{9}$$

and the following identities of measures on $\mathbb{R}_+ \times \mathbb{R}_+$,

$$x\mathbb{P}(\overleftarrow{\tau}_l^{(t)} \in \mathrm{d}x)\mathrm{d}l = x\mathbb{P}(\overrightarrow{\tau}_l^{(t)} \in \mathrm{d}x)\mathrm{d}l = l\mathbb{P}(-Y_x^{(t)} \in \mathrm{d}l)\mathrm{d}x. \tag{10}$$

Finally, we recall a useful expression for the law of $g(t)$. For all $t \in [0, 1/\mathrm{m}[$ and $\lambda \geq 0$,

$$\mathbb{E}\big(\exp\big(\lambda g(t)\big)\big) = \exp\bigg(\int_0^\infty (e^{-\lambda x} - 1)x^{-1}\mathbb{P}(Y_x^{(t)} > 0)\mathrm{d}x\bigg). \tag{11}$$

We can focus now on the evolution of the block containing 0, $\mathbf{B}_0$. First, we prove some properties of absence of memory (Section 3) : the evolution of $\mathbf{B}_0$ after time t depends from the past of this block only through $l(t)$ (Markov property). Then we focus on the left extremity : it is an additive process and we give its Lévy measure. As a consequence, we get the distribution of the instants at which the left extremity jumps (Section 4). We then derive the distribution of the remaining data which completes the description of the process of storage at the left extremity (Section 5). By taking also into account the data fallen on $\mathbf{B}_0$, we get then the evolution of $(g(t), d(t))$ (Section 6). The latter characterizes the evolution of the right extremity and the length (Section 7).

3 Markov property of $\mathbf{B}_0$

We have already proved that $\mathcal{R}(t)$ enjoys a 'spatial' regeneration property (see Proposition 3 in [1]). To study the evolution of $\mathbf{B}_0$, we need 'time' regeneration property. Here we prove that the evolution of the block containing 0 up to time t is independent of the covering outside $[g(t), d(t)]$ up to time t. In Section 5, this property will ensure that the evolution of $\mathbf{B}_0$ after time t depends from the past of this block only through $l(t)$ (Markov property).

Proposition 1. *For every $t \in [0, 1/\mathrm{m}[$, the following three processes with values in the space of subsets of $\mathbb{R}$*

. $\quad (g(t) - \mathcal{R}(s)) \cap [0, \infty[, \qquad 0 \leq s \leq t,$

. $\quad (\mathcal{R}(s) - d(t)) \cap [0, \infty[, \qquad 0 \leq s \leq t,$

. $\quad \mathcal{R}(s) \cap [g(t), d(t)], \qquad 0 \leq s \leq t,$

are independent.

Remark 1. Actually, we have the following regeneration property : $\forall t \in [0, 1/\mathrm{m}[, \forall x \in \mathbb{R}$, $\big((\mathcal{R}(s) - d_x(\mathcal{R}(t))) \cap [0, \infty[: s \in [0, t]\big)$ is independent of $\big((\mathcal{R}(s) - d_x(\mathcal{R}(t)))\cap] - \infty, 0] : s \in [0, t]\big)$ and is distributed as $\big((\mathcal{R}(s) - d_0(\mathcal{R}(t))) \cap [0, \infty[: s \in [0, t]\big)$.

This result is a direct consequence of the following lemma where we consider the point processes of files until time t at the left of/at the right of/inside $[g,d]$:

$$P_g(t) := \{(t_i, g - x_i, l_i) : t_i \le t,\ x_i < g\}, \qquad P^d(t) := \{(t_i, x_i - d, l_i) : t_i \le t,\ d < x_i\},$$
$$P_g^d(t) := \{(t_i, x_i, l_i) : t_i \le t,\ g \le x_i \le d\}.$$

Lemma 1. *For every $t \in [0, 1/\mathrm{m}[$, the point processes $P_{g(t)}(t)$, $P_{g(t)}^{d(t)}(t)$ and $P_{d(t)}(t)$ are independent.*

Proof. First we prove a weaker result, where times $(t_i)_{i\in\mathbb{N}}$ are not taken into account. Denote by $(\widetilde{Y}_x^{(t)})_{x\ge 0}$ the càdlàg version of $(Y_{-x}^{(t)})_{x\ge 0}$. This is a spectrally negative Lévy process with bounded variation, which drifts to ∞. Note that,

$$\begin{aligned} g(t) &= g_0(\mathcal{R}(t)) = \sup\{x \le 0 :\ Y_x^{(t)} = I_x^{(t)}\} \\ &= \sup\{x \le 0 :\ Y_{x^-}^{(t)} = I_0^{(t)}\} = -\inf\{x \ge 0 :\ \widetilde{Y}_x^{(t)} = \inf\{\widetilde{Y}_z^{(t)} : z \ge 0\}\}. \end{aligned}$$

Then $(\widetilde{Y}_{-g(t)+x}^{(t)} - \widetilde{Y}_{-g(t)}^{(t)})_{x\ge 0}$ is independent of $(\widetilde{Y}_x^{(t)})_{0\le x\le -g(t)}$ (decomposition of a Lévy process at its infimum [11]). Considering the locations and sizes of the jumps of these two processes yields

$$\{(g(t) - x_i, l_i) : t_i \le t,\ x_i < g(t)\} \quad \text{is independent of} \quad \{(x_i, l_i) : t_i \le t,\ g(t) \le x_i \le 0\}.$$

Adding that $\{(x_i, l_i) : t_i \le t,\ x_i > 0\}$ is independent of $\{(x_i, l_i) : t_i \le t,\ x_i \le 0\}$ and $g(t)$ is $\{(x_i, l_i) : t_i \le t,\ x_i \le 0\}$ measurable, we get

$$\{(g(t) - x_i, l_i) : t_i \le t,\ x_i < g(t)\} \quad \text{is independent of} \quad \{(x_i, l_i) : t_i \le t,\ x_i \ge g(t)\}.$$

We now extend the preceding by incorporating the times $(t_i)_{i\in\mathbb{N}}$. In this direction, we recall that if $(\widetilde{x}_i, \widetilde{l}_i)_{i\in\mathbb{N}}$ is a PPP on $\mathbb{R}\times\mathbb{R}_+$ with intensity $t\mathrm{d}x \otimes \nu(\mathrm{d}l)$ and $(\widetilde{t}_i)_{i\in\mathbb{N}}$ is an iid sequence distributed uniformly on $[0,t]$, then $\{(\widetilde{t}_i, \widetilde{x}_i, \widetilde{l}_i) : i \in \mathbb{N}\}$ is distributed as $\{(t_i, x_i, l_i) : i \in \mathbb{N}, t_i \le t\}$. Adding that $g(t)$ is $\{(x_i, l_i) : i \in \mathbb{N}, t_i \le t\}$ measurable, we get

$$\{(t_i, g(t) - x_i, l_i) : t_i \le t,\ x_i < g(t)\} \quad \text{is independent of} \quad \{(t_i, x_i, l_i) : t_i \le t,\ x_i \ge g(t)\}.$$

This ensures that $P_{g(t)}(t)$ is independent of $(P_{g(t)}^{d(t)}(t), P^{d(t)}(t))$.

One can prove similarly that $P^{d(t)}(t)$ is independent of $(P_{g(t)}(t), P_{g(t)}^{d(t)}(t))$ using that $(Y_{d(t)+x}^{(t)} - Y_{d(t)}^{(t)})_{x\ge 0}$ is independent of $(Y_x^{(t)})_{x\le d(t)}$ or Lemma 2 in [1]. □

This guarantees the absence of memory at the left of $\mathbf{B}_0(t)$. First we have :

Corollary 1. $(g(t))_{t\in[0,1/\mathrm{m}]}$ *has decreasing càdlàg paths with independent increments.*

Proof. Let $0 \leq t < t+s \leq 1/\mathrm{m}$. The increment $g(t+s) - g(t)$ just depends on $\overleftarrow{\mathcal{R}(t)}$ and the point process of files which arrive after time t at the left of $\mathbf{B}_0(t)$ $\{(t_i, x_i - g(t), l_i) : t_i > t, x_i < g(t)\}$. By the Poissonian property, these two quantities are independent and $(g(u) : u \in [0,t])$ is independent of this point process of files. Moreover $(g(u) : u \in [0,t])$ is also independent of $(g(t) - \mathcal{R}(t)) \cap [0, \infty[$ by Proposition 1. So $(g(u) : u \in [0,t])$ is independent of $g(t+s) - g(t)$. □

This explains the observation made in [1] Section 3 that the distribution of $g(t)$ is infinitively divisible (see [7] on page 174 or [13] on page 47 for details).

4 Evolution of the left extremity

Now we describe the process $(g(t))_{t\in[0,1/\mathrm{m}[}$. We know that its increments are independent and (11) specifies its marginals. We shall determine its Lévy measure and prove that its mass is finite (see [13] for terminology). This means that the instants when a file arrives at the left of $\mathbf{B}_0$ and joins this data block during its storage do not accumulate before time $1/\mathrm{m}$, even if $\bar{\nu}(0) = \infty$ (files arrive densely near the data block). Proposition 3 in [1] ensures that the first time T_1 when 0 is covered, which is also the first jump time of $(g(t))_{t\in[0,1/\mathrm{m}]}$, is uniformly distributed on $[0, 1/\mathrm{m}]$. Actually the second jump time is uniformly distributed in $[T_1, 1/\mathrm{m}]$ and so on ... More precisely, we have :

Theorem 2. *The jump times of $(g(t))_{t\in[0,1/\mathrm{m}]}$ are given by an increasing sequence $(T_i)_{i\in\mathbb{N}}$ which accumulate at $1/\mathrm{m}$. More precisely, using the convention $T_0 = 0$, it holds that for every $i \geq 1$, conditionally on $T_{i-1} = t$, T_i is independent of $(T_j)_{0\leq j\leq i-1}$ and is uniformly distributed on $[t, 1/\mathrm{m}]$.*

Then, denoting by $-G_i$ the jump of $(g(t))_{t\in[0,1/\mathrm{m}]}$ at time T_i for every $i \in \mathbb{N}$, we have

$$g(t) := -\sum_{T_i \leq t} G_i$$

where $\{(T_i, G_i) : i \in \mathbb{N}\}$ is a PPP on $[0, 1/\mathrm{m}[\times\mathbb{R}^+$ with intensity

$$\mathrm{d}t\mathrm{d}x \int_0^\infty \mathbb{P}(Y_x^{(t)} \in -\mathrm{d}l)\bar{\nu}(l).$$

In other words, $(g(t))_{t\in[0,1/\mathrm{m}]}$ is an additive process and its generating triplet is

$$\left(0, \int_0^t \mathrm{d}s \int_0^\infty \mathbb{P}(Y_x^{(s)} \in -dl)\bar{\nu}(l), 0\right).$$

In particular, the interarrival times of $\{T_i : i \in \mathbb{N}\}$ form a 'continuous uniform stick breaking sequence' (see the residual allocation model in [12] on pages 63-64) : the distribution of $\big((T_{i+1} - T_i)/\mathrm{m}\big)_{i\in\mathbb{N}}$ is the Griffiths-Engen-McCloskey distribution with parameter $(0,1)$ (i.e. rearranging these increments in the decreasing order yield the Poisson-Dirichlet distribution of parameter $(0,1)$).

Further, for every $i \in \mathbb{N}$, conditionally on $T_i = t$, the law of G_i is given by

$$\mathbb{P}(G_i \in \mathrm{d}x) = \mathrm{d}x \frac{1-\mathrm{m}t}{\mathrm{m}} \int_0^\infty \mathbb{P}(Y_x^{(t)} \in -\mathrm{d}l)\bar{\nu}(l), \tag{12}$$

and as a consequence,

$$\mathbb{E}(G_i) = \left(\frac{1}{(1-\mathrm{m}t)^2} + \frac{1}{2}\frac{\mathrm{m}}{1-\mathrm{m}t}\right) \int_0^\infty l^2 \nu(\mathrm{d}l).$$

Example 1. For the basic example ($\nu = \delta_1$), conditionally on $T_i = t$, we have,

$$\mathbb{P}(G_i \in \mathrm{d}x) = (1-t)e^{-tx}\frac{(tx)^{[x]}}{[x]!}\mathrm{d}x,$$

writing $[x] = \sup\{n \in \mathbb{N} : n \leq x\}$ and using (7).

For the proof, we need the following identity

Lemma 2. *Let $(S_t)_{t \geq 0}$ be a subordinator with no drift and Lévy tail $\bar{\mu}$. Then for all $(t,x) \in \mathbb{R}_+^2$, we have*

$$\mathbb{P}(S_t > x) = \int_0^t ds \int_0^x \mathbb{P}(S_s \in \mathrm{d}b)\bar{\mu}(x-b).$$

Proof. As S has no drift, we have for all $t > 0$ and $x > 0$,

$$S_t > x \quad \Leftrightarrow \quad \exists!\ s \in]0,t]\ :\ S_{s^-} \leq x,\ \Delta S_s > x - S_{s^-} \qquad \text{a.s.}$$

We get then, using also the compensation formula (see [2] on page 7),

$$\mathbb{P}(S_t > x) = \mathbb{E}\Big(\sum_{0<s\leq t} \mathbb{1}_{\{S_{s^-} \leq x\}} \mathbb{1}_{\{\Delta S_s > x - S_{s^-}\}}\Big) = \mathbb{E}\Big(\int_0^t \mathrm{d}s \mathbb{1}_{\{S_s \leq x\}}\bar{\mu}(x - S_s)\Big)$$

which completes the proof. One can also give an analytic proof by computing the Laplace transform of the right hand side for $q > 0$ and using Fubini :

$$\begin{aligned}
& \int_0^\infty \mathrm{d}x e^{-qx} \int_0^t \mathrm{d}s \int_0^x \mathbb{P}(S_s \in \mathrm{d}b)\bar{\mu}(x-b) \\
= & \int_0^t \mathrm{d}s \int_0^\infty \mu(\mathrm{d}y) \int_0^\infty \mathbb{P}(S_s \in \mathrm{d}b)\frac{e^{-qb} - e^{-q(b+y)}}{q} = \int_0^t \mathrm{d}s e^{-\phi(q)s} \int_0^\infty \mu(\mathrm{d}y)\frac{1-e^{-qy}}{q} \\
= & \frac{1-e^{-\phi(q)t}}{\phi(q)} \times \frac{\phi(q)}{q} = \int_0^\infty \mathrm{d}x e^{-qx}\mathbb{P}(S_t > x)
\end{aligned}$$

which proves the lemma. □

We are now able to establish Theorem 2.

Proof. We know from Corollary 1 that $(g(t))_{t\in[0,1/\mathrm{m}]}$ is an additive process. Moreover for every $x \geq 0$, $(Y_x^{(t)} + x)_{t\geq 0}$ is a subordinator with no drift and Lévy measure $x\nu$ (see (1)). So Lemma 2 ensures that

$$\begin{aligned}
\mathbb{P}(Y_x^{(t)} > 0) &= \mathbb{P}(Y_x^{(t)} + x > x) \\
&= \int_0^t \mathrm{d}s \int_0^x \mathbb{P}(Y_x^{(s)} + x \in \mathrm{d}b) x\bar{\nu}(x-b) \\
&= \int_0^t \mathrm{d}s \int_0^\infty \mathbb{P}(Y_x^{(s)} \in -\mathrm{d}l) x\bar{\nu}(l).
\end{aligned}$$

Using (11), we get

$$\mathbb{E}\big(\exp\big(\lambda g(t)\big)\big) = \exp\Big(\int_0^\infty \mathrm{d}x (e^{-\lambda x} - 1)\int_0^t \mathrm{d}s \int_0^\infty \mathbb{P}(Y_x^{(s)} \in -\mathrm{d}l)\bar{\nu}(l)\Big).$$

So $(g(t))_{t\in[0,1/\mathrm{m}]}$ is an additive process with generating triplet

$$\Big(0, \int_0^t \mathrm{d}s \int_0^\infty \mathbb{P}(Y_x^{(s)} \in -\mathrm{d}l)\bar{\nu}(l), 0\Big)$$

using Definition 8.2 and Theorem 9.8 in [13]. This characterizes the distribution of $(g(t))_{t\in[0,1/\mathrm{m}]}$ (by Theorem 9.8 in [13]) and proves that $\{(T_i, G_i) : i \in \mathbb{N}\}$ is a PPP on $[0, 1/\mathrm{m}[\times\mathbb{R}^+$ with intensity $\mathrm{d}t\mathrm{d}x \int_0^\infty \mathbb{P}(Y_x^{(t)} \in -\mathrm{d}l)\bar{\nu}(l)$. One can also compute the distribution of $g(t+s) - g(t)$ using the independence of increments and (11) : this proves that that $g(.)$ is the sum of jumps given by a PPP.

By projection, $\{T_i : i \in \mathbb{N}\}$ is a PPP on $[0, 1/\mathrm{m}[$ with intensity $\mathrm{m}(1-\mathrm{m}t)^{-1}\mathrm{d}t$. Indeed, for every $t \in [0, 1/\mathrm{m}[$,

$$\begin{aligned}
\int_0^\infty \mathrm{d}x \int_0^\infty \mathbb{P}(Y_x^{(t)} \in -\mathrm{d}l)\bar{\nu}(l) &= \int_0^\infty \mathbb{P}(\overrightarrow{\tau}_l^{(t)} \in \mathrm{d}x) \int_0^\infty \mathrm{d}l \frac{x}{l}\bar{\nu}(l) \quad \text{using (10)} \\
&= \int_0^\infty \mathrm{d}l \frac{\mathbb{E}(\overrightarrow{\tau}_l^{(t)})\bar{\nu}(l)}{l} \\
&= \mathbb{E}(\overrightarrow{\tau}_1^{(t)}) \int_0^\infty \bar{\nu}(l)\mathrm{d}l \\
&= \frac{\mathrm{m}}{1-\mathrm{m}t} \quad \text{using (9).}
\end{aligned}$$

Thus, writing $N_t^{t'} := \mathrm{card}\{i \in \mathbb{N} : T_i \in]t, t']\}$, we have $N_0^t < \infty$ a.s. for every $t \in [0, 1/\mathrm{m}[$. We we can then sort the times T_i and we have

$$\mathbb{P}(T_{i+1} > t' \mid T_i = t) = \mathbb{P}(N_t^{t'} = 0) = \exp\big(-\int_t^{t'} \mathrm{d}s \frac{\mathrm{m}}{1-\mathrm{m}s}\big) = \frac{1-\mathrm{m}t'}{1-\mathrm{m}t},$$

meaning that T_{i+1} is uniformly distributed in $[T_i, 1/\mathrm{m}]$. The independence is a consequence of the Poissonian property of $\{T_i : i \in \mathbb{N}\}$ and we get the theorem.

Finally, this proves (12) and for every $i \in \mathbb{N}$, conditionally on $T_i = t$, we get

$$\begin{aligned} \mathbb{E}(G_i) &= \frac{1-\mathrm{m}t}{\mathrm{m}} \int_0^\infty \mathrm{d}l \frac{\mathbb{E}([\overrightarrow{\tau}_l^{(t)}]^2)\bar{\nu}(l)}{l} \quad \text{using again (10)} \\ &= \frac{1-\mathrm{m}t}{\mathrm{m}} \int_0^\infty \mathrm{d}l \bar{\nu}(l) \big(l \big(\frac{\mathrm{m}}{1-\mathrm{m}t} \big)^2 + \frac{\int_0^\infty l^2 \nu(\mathrm{d}l)}{(1-\mathrm{m}t)^3} \big) \end{aligned}$$

since $[\kappa^{(t)}]'(0)$ is given by (9) and $[\kappa^{(t)}]''(0)$ is given by Proposition 4 in [1]. □

5 The process of remaining data

We still consider the files which arrive at the left of $\mathbf{B}_0$, the block containing 0, and cannot be entirely stored at the left of this block (see Figure 2). Such events occur at the jump times of $(g(t))_{t\in[0,1/\mathrm{m}[}$, that is at time T_i. We focus here on the portions of these files which cannot be stored at the left of $\mathbf{B}_0$ and are shifted to the right of $\mathbf{B}_0(T_i-)$ to find a free space. They are called remaining data and denoted by R_i. Thus R_i is the quantity of data which arrives at the left of $\mathbf{B}_0$ at time T_i and is stored at the right of $\mathbf{B}_0$. Then it is also the quantity of data over $g(T_{i-1}-)$ at time T_i (see Section 2.1 in [1] for details) and it is given by

$$\forall i \geq 1, \qquad R_i := Y^{(T_i)}_{g(T_{i-1}-)} - I^{(T_i)}_{g(T_{i-1}-)}.$$

We aim at determining the distribution of $\{(T_i, G_i, R_i) \ : \ i \in \mathbb{N}\}$ which is the key to the characterization of the jumps of $(g(t), d(t))_{t\in[0,1/\mathrm{m}[}$. In that view, we need to describe the arrival of files which induce the jumps (G_i, R_i). So we consider the half hardware at the left of $g(t)$, which we turn over, so that it is now identified with $\mathbb{R}^+$ and its free space is given by $\overleftarrow{\mathcal{R}(t)}$ (see Section 2). The size of free space and the first free plots of this half hardware are given by the processes $(L_x^{(t)})_{x\geq 0}$ and $(D_x^{(t)})_{x\geq 0}$ defined by

$$\forall t \in [0, 1/\mathrm{m}[, \ \forall x \geq 0, \qquad L_x^{(t)} = | \ \overleftarrow{\mathcal{R}(t)} \cap [0, x] \ |, \qquad D_x^{(t)} = \inf\{y > x : y \in \overleftarrow{\mathcal{R}(t)}\}.$$

When at time t, a file of length l arrives at location $-x + g(t-)$ on the hardware (i.e. at location x on the half hardware), it yields a jump of $g(.)$ if the free space $L_x^{(t-)}$ between $-x + g(t-)$ and $g(t-)$ is less than l. Then the quantity of remaining data is $l - L_x^{(t-)}$ and the jump of the left extremity is $D_x^{(t-)}$ (see Figure 2). So we naturally introduce the measure $\rho^{(t)}$ on $\mathbb{R}_+^2$ defined by

$$\rho^{(t)}(\mathrm{d}y\mathrm{d}z) := \int_0^\infty \mathrm{d}x \int_0^\infty \nu(\mathrm{d}l)\mathbb{P}(D_x^{(t)} \in \mathrm{d}y, l - L_x^{(t)} \in \mathrm{d}z).$$

In forthcoming Lemma 3, we give a useful alternative expression of $\rho^{(t)}$. This measure gives the intensity of the point process $\{(T_i, G_i, R_i) : i \in \mathbb{N}\}$, as stated by the following result.

Theorem 3. *$\{(T_i, G_i, R_i) : i \in \mathbb{N}\}$ is a PPP on $[0, 1/\mathrm{m}[\times\mathbb{R}_+^2$ with intensity $\mathrm{d}t\rho^{(t)}(\mathrm{d}y\mathrm{d}z)$.*

A remarkable consequence is that $(R_i)_{i\in\mathbb{N}}$ is an iid sequence : whereas the rate at which jumps occur increases as time gets closer to $1/\mathrm{m}$, the quantity of remaining data keeps the same distribution.

Corollary 2. *$\{(T_i, R_i) : i \in \mathbb{N}\}$ is a PPP on $[0, 1/\mathrm{m}[\times\mathbb{R}^+$ with intensity $\mathrm{d}t\mathrm{d}z\frac{\bar{\nu}(z)}{1-\mathrm{m}t}$. In other words, $(R_i)_{i\in\mathbb{N}}$ is iid, independent of $(T_i)_{i\in\mathbb{N}}$ and its distribution is given by :*

$$\mathbb{P}(R_i \in \mathrm{d}z) = \mathrm{m}^{-1}\bar{\nu}(z)\mathrm{d}z, \quad z \geq 0.$$

Example 2. Using the expression of $\rho^{(t)}$ given by Lemma 3 below, the expressions (23) and (24) in [1] yield an expression of $\rho^{(t)}$ for the basic example and the gamma distribution which is quite heavy and not mentioned here. Nonetheless the quantity of remaining data can be often calculated explicitly. For the basic example ($\nu = \delta_1$), the remaining data are uniform random variables on $[0, 1]$. For the exponential distribution ($\nu(\mathrm{d}l) = \mathbb{1}_{\{l\geq 0\}}e^{-l}\mathrm{d}l$), the remaining data are also exponentially distributed.

The proofs of these results are organized as follows.
First, in Lemma 3, we give a more explicit expression of $\rho^{(t)}$ which will be useful for the proofs and will enable us to derive Corollary 2 from Theorem 3.
Second, we prove that $\rho^{(t)}$ gives the intensity of the point process $\{(T_i, G_i, R_i) : i \in \mathbb{N}\}$ (Lemma 4). That is for every $t \in [0, 1/\mathrm{m}[$ and $A =]a_1, b_1]\times]a_2, b_2] \subset \mathbb{R}_+^2$, we have :

$$\lim_{h\to 0} \frac{\mathbb{P}(\exists i \in \mathbb{N} : T_i \in]t, t+h],\ (G_i, R_i) \in A)}{h} = \rho^{(t)}(A).$$

The lowerbound appears naturally by considering the arrival of one single file independently of the past which induces a jump of the left extremity, as described at the beginning of this section (see also Figure 2). However, in the case $\bar{\nu}(0) = \infty$, some jumps of the left extremity could be due to the successive arrival of many files during a short time interval $]t, t+h]$. Thanks to Theorem 2, we already know the rate at which jumps occur (i.e. the total intensity). This will give us the upperbound.
Finally, we prove that the point process $\{(T_i, G_i, R_i) : i \in \mathbb{N}\}$ enjoys a memoryless property (Lemma 5), which is a direct consequence of results of Section 3. We get then the complete description of this point process, which enables us to prove Theorem 3. Corollary 2 follows by integrating $\rho^{(t)}$ with respect to the first coordinate.

Recall the notation in Theorem 1 and (6).

Lemma 3. *For every $t \in [0, 1/\mathrm{m}[$, the measure $\rho^{(t)}(\mathrm{d}y\mathrm{d}z)$ can also be expressed as*

$$\mathrm{d}z\int_z^\infty \nu(\mathrm{d}l)\Big(\mathbb{P}(\overleftarrow{\tau}^{(t)}_{l-z} \in \mathrm{d}y) + \int_0^y \mathbb{P}(\overleftarrow{\tau}^{(t)}_{l-z} \in \mathrm{d}x)(y-x)\Pi^{(t)}(\mathrm{d}y - x)\Big)$$

$$= \int_z^\infty \nu(\mathrm{d}l)(l-z)\Big(y^{-1}\mathrm{d}y\mathbb{P}(Y_y^{(t)} + l \in \mathrm{d}z) + \int_0^y \mathbb{P}(Y_x^{(t)} + l \in \mathrm{d}z)(yx^{-1} - 1)\Pi^{(t)}(\mathrm{d}y - x)\Big)$$

Proof. By Lemma 1.11 in Chapter 1 of [4] applied to $(\overleftarrow{\tau}^{(t)}_x)_{x\geq 0}$, we have for all $a, b \geq 0$ and $q > 0$ (t is fixed and omitted in the notation),

$$\int_0^\infty \mathrm{d}x e^{-qx}\mathbb{E}(\exp(-bL_x - aD_x)) = \frac{\kappa(a+q)-\kappa(a)}{q(\kappa(a+q)+b)}.$$

Letting $q \to 0$, we get

$$\int_0^\infty \mathrm{d}x\mathbb{E}(\exp(-bL_x - aD_x)) = \frac{\kappa'(a)}{\kappa(a)+b} = \int_0^\infty \mathrm{d}z e^{-bz}\kappa'(a)e^{-\kappa(a)z}.$$

From $\kappa'(a) = \int_0^\infty e^{-ay}(\delta_0(\mathrm{d}y) + y\Pi(\mathrm{d}y))$ and $e^{-\kappa(a)z} = \int_0^\infty e^{-ay}\mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}y)$, we deduce

$$\int_0^\infty \mathrm{d}x\mathbb{E}(\exp(-bL_x - aD_x)) = \int_0^\infty \mathrm{d}z \int_0^\infty \gamma_z(\mathrm{d}y)e^{-bz-ay}, \tag{13}$$

where γ_z is the convolution of $\delta_0(\mathrm{d}y) + y\Pi(\mathrm{d}y)$ and $\mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}y)$. Thus,

$$\begin{aligned} \gamma_z(\mathrm{d}y) &= \int_0^y \mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}x)(\delta_0(\mathrm{d}y - x) + (y-x)\Pi(\mathrm{d}y - x)) \\ &= \mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}y) + \int_0^y \mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}x)(y-x)\Pi(\mathrm{d}y - x). \end{aligned}$$

And the identification of Laplace transforms in (13) entails that

$$\int_0^\infty \mathrm{d}x\mathbb{P}(L_x \in \mathrm{d}z, D_x \in \mathrm{d}y) = \mathrm{d}z\big(\mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}y) + \int_0^y \mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}x)(y-x)\Pi(\mathrm{d}y-x)\big), \tag{14}$$

which proves the first identity of the lemma integrating with respect to l. Using (10) gives the second one. $\square$

Remark 2. A recent work of Winkel (Theorem 1 in [14]) enables to calculate differently the law of $\mathbb{P}(L_x \in \mathrm{d}z, D_x \in \mathrm{d}y)$ (L_x corresponds to T_x in [14] and D_x to $X(T_{x-}) + \Delta_x$) :

$$\int_0^\infty \mathrm{d}x\mathbb{P}(L_x \in \mathrm{d}z, D_x \in \mathrm{d}y) = \mathrm{d}y\mathbb{P}(H_y \in \mathrm{d}z) + \mathrm{d}z \int_0^\infty \mathbb{P}(\overleftarrow{\tau}_x \in \mathrm{d}x)(y-x)\Pi(\mathrm{d}y - x),$$

where $H_x = \inf\{a \geq 0, \overleftarrow{\tau}_a = x\}$. Then observe that the measures on $\mathbb{R}_+^2$ $\mathrm{d}y\mathbb{P}(H_y \in \mathrm{d}z)$ and $\mathrm{d}z\mathbb{P}(\overleftarrow{\tau}_z \in \mathrm{d}y)$ coincide by computing their Laplace transform using (4) in [14]. This proves (14).

Second, for every Borel set B of $[0, 1/\mathrm{m}[\times\mathbb{R}_+^2$, we define $N_B := \mathrm{card}\{i \in \mathbb{N} : (T_i, G_i, R_i) \in B\}$ and we say that A is a rectangle of $D \subset \mathbb{R}^{\mathrm{d}}$ if A is a subset of D of the form

$$\{x = (x_1, x_2, .., x_{\mathrm{d}}), a_1 < x_1 \leq b_1, .., a_{\mathrm{d}} < x_{\mathrm{d}} \leq b_{\mathrm{d}}\}.$$

Then, we have

Lemma 4. *For all $t \in [0, 1/\mathrm{m}[$ and A rectangle of $\mathbb{R}_+^2$, we have :*

$$\lim_{h\to 0} \frac{\mathbb{P}(N_{]t,t+h]\times A} \geq 1)}{h} = \rho^{(t)}(A).$$

Proof. First we prove the lowerbound. Second, we check that the convergence holds for $A=\mathbb{R}_+^2$.

• Let $\epsilon>0$, $A=]a,b]\times]c,d]$ and work conditionally on $\overleftarrow{\mathcal{R}(t)}$. We consider a file labelled i which arrives at time $t_i\in]t,t+h]$ at location $x_i<g(t)$. We put $\widetilde{x}_i:=g(t)-x_i\geq 0$ the arrival point on the half line at the left of $g(t)$ and require that

$$l_i-L^{(t)}_{\widetilde{x}_i}\in]c,d-\epsilon],\ D^{(t)}_{\widetilde{x}_i}\in]a,b-\epsilon],\ |L^{(t_i-)}_{\widetilde{x}_i}-L^{(t)}_{\widetilde{x}_i}|\leq\epsilon,\ |D^{(t_i-)}_b-D^{(t)}_b|\leq\epsilon.$$

Then file i verifies

$$l_i-L^{(t_i-)}_{\widetilde{x}_i}\in]c,d],\ D^{(t_i-)}_{\widetilde{x}_i}\in]a,b].$$

So this file induces a jump of the left extremity and $N_{]t,t+h]\times A}\geq 1$ (see the beginning of this section or Figure 2 for details) and we get the lowerbound :

$$\begin{aligned}
&\mathbb{P}\big(N_{]t,t+h]\times A}\geq 1\ \mid\ \overleftarrow{\mathcal{R}(t)}\big)\\
&\quad\geq\ \mathbb{P}\big(\exists i\in\mathbb{N}\ :\ t_i\in]t,t+h],\ l_i-L^{(t)}_{\widetilde{x}_i}\in]c,d-\epsilon],\ D^{(t)}_{\widetilde{x}_i}\in]a,b-\epsilon],\\
&\qquad\quad |L^{(t_i-)}_{\widetilde{x}_i}-L^{(t)}_{\widetilde{x}_i}|\leq\epsilon,\ |D^{(t_i-)}_b-D^{(t)}_b|\leq\epsilon\ \mid\ \overleftarrow{\mathcal{R}(t)}\big)\\
&\quad\geq\ A_t(h).B_t(h)
\end{aligned}\tag{15}$$

where

$$\begin{aligned}
A_t(h)&:=\mathbb{P}\big(\exists i\in\mathbb{N}:\ t_i\in]t,t+h],l_i-L^{(t)}_{\widetilde{x}_i}\in]c,d-\epsilon],D^{(t)}_{\widetilde{x}_i}\in]a,b-\epsilon]\ \mid\ \overleftarrow{\mathcal{R}(t)}\big),\\
B_t(h)&:=\mathbb{P}\big(\sup_{t'\in[t,t+h]}\{|L^{(t')}_b-L^{(t)}_b|\}\leq\epsilon,\ \sup_{t'\in[t,t+h]}\{|D^{(t')}_b-D^{(t)}_b|\}\leq\epsilon\ \mid\ \overleftarrow{\mathcal{R}(t)}\big).
\end{aligned}$$

1) By Theorem 2, $\mathbb{P}(N_t^{t+h}\neq 0)\xrightarrow{h\to 0}0$ so a.s for h small enough, $g(t+h)=g(t)$. Then, using the Hausdorff metric on $\mathbb{R}_+$ (denoted by $\mathcal{H}(\mathbb{R}_+)$ in Section 2 in [1]), we have

$$\overleftarrow{\mathcal{R}(t+h)}\xrightarrow{h\to 0}\overleftarrow{\mathcal{R}(t)}\quad\text{a.s.}$$

Then $B_t(h)$ converges a.s. to 1 as h tends to 0.

2) As $\{(t_i,\widetilde{x}_i,l_i):i\in\mathbb{N},t_i\in]t,t+h],x_i<g(t)\}$ is a PPP on $]t,t+h]\times\mathbb{R}_+^2$ with intensity $\mathrm{d}t\otimes\mathrm{d}x\otimes\nu(\mathrm{d}l)$ independent of $\overleftarrow{\mathcal{R}(t)}$,

$$A_t(h)=1-\exp\big(-h\int_0^\infty\mathrm{d}x\int_0^\infty\nu(\mathrm{d}l)\mathbb{1}_{\{l-L^{(t)}_x\in]c,d-\epsilon],D^{(t)}_x\in]a,b-\epsilon]\}}\big)\qquad\text{a.s.}$$

This term is a.s. equivalent when h tends to 0 to

$$h\int_0^\infty\mathrm{d}x\int_0^\infty\nu(\mathrm{d}l)\mathbb{1}_{\{l-L^{(t)}_x\in]c,d-\epsilon],D^{(t)}_x\in]a,b-\epsilon]\}}.$$

Then, letting $h\to 0$ in (15), 1) and 2) give

$$\liminf_{h\to 0}\frac{\mathbb{P}\big(N_{]t,t+h]\times A}\geq 1\ \mid\ \overleftarrow{\mathcal{R}(t)}\big)}{h}\geq\int_0^\infty\mathrm{d}x\int_0^\infty\nu(\mathrm{d}l)\mathbb{1}_{\{l-L^{(t)}_x\in]c,d-\epsilon],D^{(t)}_x\in]a,b-\epsilon]\}}\quad\text{a.s.}$$

Integrating this inequality and using Fatou's lemma yield

$$\begin{aligned}\liminf_{h\to 0}\frac{\mathbb{P}\big(N_{]t,t+h]\times A}\geq 1\big)}{h}\geq&\mathbb{E}\Big(\int_0^\infty \mathrm{d}x\int_0^\infty \nu(\mathrm{d}l)\mathbb{1}_{\{l-L_x^{(t)}\in]c,d-\epsilon],D_x^{(t)}\in]a,b-\epsilon]\}}\Big)\\ \geq&\rho^{(t)}(]a,b-\epsilon]\times]c,d-\epsilon]).\end{aligned}$$

As $\rho^{(t)}(]a,b]\times\{d\}\cup\{b\}\times]c,d])=0$ (use the two equalities of Lemma 3), we get letting ϵ tend to 0 :

$$\liminf_{h\to 0}\frac{\mathbb{P}\big(N_{]t,t+h]\times A}\geq 1\big)}{h}\geq \rho^{(t)}(A).$$

• We derive the upperbound from Theorem 2. First,

$$\frac{\mathbb{P}\big(N_{]t,t+h]\times\mathbb{R}_+^2}\geq 1\big)}{h}=\frac{\mathbb{P}(\exists i\in\mathbb{N}\ :\ T_i\in]t,t+h])}{h}\overset{h\to 0}{\longrightarrow}\frac{\mathrm{m}}{1-\mathrm{m}t}.$$

and identity (17) below gives

$$\rho^{(t)}(\mathbb{R}_+^2)=\frac{\mathrm{m}}{1-\mathrm{m}t}.$$

So we just need to prove the following result : Let $(\mu_n)_{n\in\mathbb{N}}$ and μ be finite measures on $\mathbb{R}_+^2$ such that for every A rectangle of $\mathbb{R}_+^2$: $\liminf_{n\to\infty}\mu_n(A)\geq\mu(A)$ and $\lim_{n\to\infty}\mu_n(\mathbb{R}_+^2)=\mu(\mathbb{R}_+^2)$. Then for every A rectangle of $\mathbb{R}_+^2$, $\lim_{n\to\infty}\mu_n(A)=\mu(A)$.
In that view, suppose there exist a rectangle A, $\epsilon>0$ and a sequence of integers k_n such that $\mu_{k_n}(A)\geq\mu(A)+\epsilon$. Choose B union of disjoint rectangles all disjoint from A such that $\mu(B\cup A)\geq\mu(\mathbb{R}_+^2)-\epsilon/2$. Then,

$$\liminf_{n\to\infty}\mu_{k_n}(\mathbb{R}_+^2)\geq\liminf_{n\to\infty}\mu_{k_n}(A\cup B)\geq\mu(A)+\epsilon+\mu(B)\geq\mu(\mathbb{R}_+^2)+\epsilon/2,$$

which is a contradiction with $\lim_{n\to\infty}\mu_n(\mathbb{R}_+^2)=\mu(\mathbb{R}_+^2)$. $\square$

To prove the theorem, it remains to prove the absence of memory.

Lemma 5. *Let $t\in[0,1/\mathrm{m}[$, then $\big\{(T_i,G_i,R_i)\ :\ i\in\mathbb{N},\ T_i\leq t\big\}$ is independent of $\big\{(T_i,G_i,R_i)\ :\ i\in\mathbb{N},\ T_i>t\big\}$.*

Proof. First $\big\{(T_i,G_i,R_i)\ :\ T_i\leq t\big\}$ is given by $\big\{(t_i,l_i,x_i):t_i\leq t,x_i\in[g(t),d(t)]\big\}$. Moreover $\big\{(T_i,G_i,R_i)\ :\ T_i>t\big\}$ depends on $(\mathcal{R}(t)-g(t))\cap]-\infty,0]$ and $\big\{(t_i,x_i-g(t),l_i)\ :\ t_i>t,\ x_i<g(t)\big\}$ which are independent. Moreover $(\mathcal{R}(t)-g(t))\cap]-\infty,0]$ is independent of $\big\{(t_i,l_i,x_i):t_i\leq t,x_i\in[g(t),d(t)]\big\}$ by Lemma 1 and so is $\big\{(t_i,x_i-g(t),l_i)\ :\ t_i>t,\ x_i<g(t)\big\}$ by Poissonian property. This proves the result. $\square$

We can now prove the theorem and its corollary.

Proof of Theorem 3. We prove now that for every B finite union of disjoint rectangles of $[0,1/\mathrm{m}[\times\mathbb{R}_+^2$:

$$\mathbb{P}(N_B=0)=e^{-\gamma(B)},\quad\text{where }\ \gamma(\mathrm{d}t\mathrm{d}y\mathrm{d}z)=\mathrm{d}t\rho^{(t)}(\mathrm{d}y\mathrm{d}z).\tag{16}$$

As γ is non atomic (use Lemma 3), this will ensure that $\{(T_i, G_i, R_i) : i \in \mathbb{N}\}$ is a PPP with intensity γ (use Renyi's Theorem [10]).

Let $t \in [0, 1/\mathrm{m}[$ and A a finite union of rectangles of $\mathbb{R}_+^2$. We consider $H(s) := \mathbb{P}(N_{]t,t+s]\times A} = 0)$ for $s \in [0, 1/\mathrm{m} - t[$. Lemma 5 entails that

$$H(s+h) = \mathbb{P}(N_{]t,t+s]\times A} = 0)\mathbb{P}(N_{]t+s,t+s+h]\times A} = 0) = H(s)\mathbb{P}(N_{]t+s,t+s+h]\times A} = 0).$$

We write $A = \sqcup_{i=1}^N A_i$ where A_i rectangle of $\mathbb{R}_+^2$. Theorem 2 and Lemma 4 ensure respectively that for all $1 \le i, j \le N$ such that $i \neq j$:

$$\lim_{h\to 0} \frac{\mathbb{P}(N_{]t,t+h]\times A_i} \ge 1,\ N_{]t,t+h]\times A_j} \ge 1)}{h} = 0 \quad ; \quad \lim_{h\to 0} \frac{\mathbb{P}(N_{]t,t+h]\times A_i} \ge 1)}{h} = \rho^{(t)}(A_i).$$

Then

$$\lim_{h\to 0} \frac{\mathbb{P}(N_{]t,t+h]\times A} \ge 1)}{h} = \sum_{i=1}^N \lim_{h\to 0} \frac{\mathbb{P}(N_{]t,t+h]\times A_i} \ge 1)}{h} = \rho^{(t)}(A),$$

and the derivative of H is given by

$$\lim_{h\to 0} \frac{H(s+h) - H(s)}{h} = H(s) \lim_{h\to 0} \frac{1 - \mathbb{P}(N_{]t+s,t+s+h]\times A} = 0)}{h} = H(s)\rho^{(t+s)}(A).$$

Thus $H(s)$ satisfies a differential equation of order 1 and we get (16) for $B =]t, t+s] \times A$.

$$H(s) = \exp\big(-\int_0^s \mathrm{d}u\rho^{(t+u)}(A)\big) = \exp\big(-\int_t^{t+s} \mathrm{d}u\rho^{(u)}(A)\big) = e^{-\gamma(]t,t+s]\times A)}$$

Using again Lemma 5 and additivity of measures proves (16) for every B finite union of rectangles of $[0, 1/\mathrm{m}[\times\mathbb{R}^+ \times \mathbb{R}^+$. $\square$

Proof of Corollary 2. As projection of the PPP $\{(T_i, G_i, R_i) : i \in \mathbb{N}\}$, $\{(T_i, R_i) : i \in N\}$ is a PPP with intensity $\mathrm{d}t \int_{y\in[0,\infty]} \rho^{(t)}(\mathrm{d}y\mathrm{d}z)$. By Lemma 3, we have :

$$\begin{aligned}
\int_{y\in[0,\infty]} \rho^{(t)}(\mathrm{d}y\mathrm{d}z) &= \mathrm{d}z\big(\bar{\nu}(z) + \int_z^\infty \nu(\mathrm{d}l) \int_0^\infty \mathbb{P}(\overleftarrow{\tau}_{l-z}^{(t)} \in \mathrm{d}x) \int_x^\infty \Pi^{(t)}(dy-x)(y-x)\big) \\
&= \mathrm{d}z\bar{\nu}(z)(1 + \int_0^\infty \Pi(\mathrm{d}y)y) \\
&= \mathrm{d}z\frac{\bar{\nu}(z)}{1-\mathrm{m}t} \qquad \text{by (9)} \qquad (17)
\end{aligned}$$

which gives the intensity of $\{(T_i, R_i) : i \in \mathbb{N}\}$. In other words, $(R_i)_{i\in\mathbb{N}}$ is an iid sequence independent of $(T_i)_{i\in\mathbb{N}}$ such that $\mathbb{P}(R_i \in dz) = \mathrm{m}^{-1}\bar{\nu}(z)\mathrm{d}z,\ (z \ge 0)$. $\square$

6 Evolution of $\mathbf{B}_0$

The processes $(g(t))_{t\in[0,1/\mathrm{m}]}$ and $(d(t))_{t\in[0,1/\mathrm{m}]}$ of the left and the right extremities of $\mathbf{B}_0$ have a quite different evolution, even though their one-dimensional distributions coincide. The process $(d(t))_{t\in[0,1/\mathrm{m}]}$ jumps each time $(g(t))_{t\in[0,1/\mathrm{m}]}$ jumps and each time a file arrives on $\mathbf{B}_0$. More precisely, there are two kinds of jumps of $(\mathbf{B}_0(t))_{t\in[0,1/\mathrm{m}]}$ corresponding respectively to :
- files which arrive at the left of $\mathbf{B}_0$ and cannot be entirely stored at its left (recall the previous section). These files induce the jumps $(-G_i, D_i)$ of the extremities of $\mathbf{B}_0$ at time T_i independently of the past (see Figure 2).
- files which arrive on $\mathbf{B}_0$. These files induce jumps of the right extremity $d(.)$ only, with total rate equal to $l(t)\bar{\nu}(0)$ (see Figure 3). This rate is infinite when $\bar{\nu}(0)=\infty$. Observe also that the jumps depend from the past of $\mathbf{B}_0$ through the value of the length $l(t)$.
Note that a file which arrives at the left of $\mathbf{B}_0(t-)$ at time t with remaining data of size R induces the same jump of the right extremity as a file of size R which arrives on $\mathbf{B}_0(t-)$ at time t. Obviously, the other files (files which are entirely stored at the left of $\mathbf{B}_0$ or which arrive at the right of $\mathbf{B}_0$) do not yield a jump of $\mathbf{B}_0$.

Thus, we define
$$D_i := d(T_i) - d(T_i^-)$$
and we decompose the process $(g(t), d(t))_{t\in[0,1/\mathrm{m}[}$ into two processes $(C^1(t))_{t\in[0,1/\mathrm{m}[}$ and $(C^2(t))_{t\in[0,1/\mathrm{m}[}$, which give the variation of the extremities of $\mathbf{B}_0$ respectively at times $(T_i)_{i\in\mathbb{N}}$ (due to the arrival of a file at the left of $g(t)$) and between successive times $(T_i)_{i\in\mathbb{N}}$ (due to the arrival of files on $\mathbf{B}_0(t)$). That is, for every $t\in[0,1/\mathrm{m}[$,

$$C^1(t) := \sum_{T_i\le t}(-G_i, D_i), \qquad C^2(t) := \big(0, \sum_{\substack{0\le s\le t\\ s\notin\{T_i : i\in\mathbb{N}\}}} \Delta d(s)\big),$$

$$(g(t), d(t)) = C^1(t) + C^2(t).$$

First, we specify the distribution of $(C^1(t))_{t\in[0,1/\mathrm{m}]}$ (see below for the proofs).

Proposition 2. *The point process* $\big\{(T_i, G_i, D_i) : i\in\mathbb{N}\big\}$ *is a PPP on* $[0,1/\mathrm{m}]\times\mathbb{R}_+^2$ *with intensity* $\mathrm{d}t\mu^{(t)}(\mathrm{d}y\mathrm{d}x)$*, where*

$$\mu^{(t)}(\mathrm{d}y\mathrm{d}x) = \int_0^\infty \rho^{(t)}(\mathrm{d}y\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}x).$$

We can now specify the distribution of the process $(g(t), d(t))_{t\in[0,1/\mathrm{m}[}$ as follows.

Theorem 4. *$(g(t),d(t))_{t\in[0,1/\mathrm{m}[}$ is a pure jump Markov process equal to $(C^1(t)+C^2(t))_{t\in[0,1/\mathrm{m}]}$ such that for all $0\leq t\leq t+s\leq 1/\mathrm{m}$,*

(i) $C^1(t+s)-C^1(t)$ is independent of $(g(u),d(u))_{u\in[0,t]}$.

(ii) Conditionally on $l(t)=l$, $C^2(t+s)-C^2(t)$ is independent of $(g(u),d(u))_{u\in[0,t]}$. Conditionally also on $T_i\leq t\leq t+s<T_{i+1}$ for some $i\in\mathbb{N}$:

$$C^2(t+s)-C^2(t)\stackrel{d}{=}(0,\overrightarrow{\tau}^{(t+s)}_{S_{sl}}),$$

where $(S_x)_{x\geq 0}$ is a subordinator with no drift and Lévy measure ν, which is independent of $(\overrightarrow{\tau}^{(t+s)}_x)_{x\geq 0}$.

We recall that vague convergence of measures on A is the convergence of the integrals of measures against continuous functions with compact support in A. The jump rate of $(g(t),d(t))_{t\in[0,1/\mathrm{m}[}$ is then given by :

Corollary 3. *If $t\in[0,1/\mathrm{m}[$, we have the following vague convergence of measures on $[0,\infty[\times]0,\infty[$ when h tends to 0 :*

$$h^{-1}\mathbb{P}(g(t)-g(t+h)\in \mathrm{d}y,\ d(t+h)-d(t)\in \mathrm{d}x\ \mid\ l(t)=l)\ \stackrel{w}{\Longrightarrow}$$

$$\mu^{(t)}(\mathrm{d}y\mathrm{d}x)+l\delta_0(\mathrm{d}y)\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}^{(t)}_z\in \mathrm{d}x).$$

We begin with two lemmas which state the independences needed for the proofs.

Lemma 6. *$\{(T_i,G_i,D_i):i\in\mathbb{N},\ T_i>t\}$ is independent of $(g(u),d(u))_{u\in[0,t]}$.*

Proof. Using (18) below, we see that $\{(T_i,G_i,D_i):i\in\mathbb{N},\ T_i>t\}$ is given by

$$\{(T_i,G_i,R_i):i\in\mathbb{N},T_i>t\}\quad\text{and}\quad(\overrightarrow{\mathcal{R}}(s))_{s>t}.$$

These quantities depend from the past through $(\overleftarrow{\mathcal{R}(t)},\overrightarrow{\mathcal{R}(t)})$ which is independent of $(g(u),d(u))_{u\in[0,t]}$ by Proposition 1. □

Lemma 7. *Let $i\in\mathbb{N}$ and $0\leq t'<t\leq 1/\mathrm{m}$. Conditionally on $T_{i-1}=t'$ and $T_i=t$, $(\overrightarrow{\mathcal{R}(u)})_{u\in[t',t[}$ is independent of the point process $P_{g(t')}(t)$.*

Proof. Conditioning by $T_{i-1}=t'$ and $T_i=t$ ensures that all the data arrived at the left of $g(t')$ during the time interval $[t',t[$ are stored at the left of $g(t')$. So $(\overrightarrow{\mathcal{R}(u)})_{u\in[t',t[}$ depdns only on the point process $P^{d(t')}_{g(t')}(t)\cup P^{d(t')}(t)$ which is independent of $P_{g(t')}(t)$ by Lemma 1. □

Proof of Proposition 2. At time T_i, the quantity of remaining data R_i is stored at the right of $\mathbf{B}_0(T_i-)$. It induces a jump $D_i = d(T_i) - d(T_i-)$ of the right extremity which is equal to R_i plus the sum of the lengths of blocks at the right of $\mathbf{B}_0(T_i-)$ which are reached during the storage of these data (see Figure 2). More precisely :

$$\begin{aligned} D_i &= \inf\{x \geq 0, \mid \mathcal{R}(T_i-) \cap [d(t), d(t)+x[| = R_i\} \\ &= \inf\{x \geq 0, \mid \overrightarrow{\mathcal{R}(T_i-)} \cap [0,x] \mid = R_i\} \\ &= \overrightarrow{\tau}^{(T_i-)}_{R_i}, \end{aligned} \tag{18}$$

by definition of $\overrightarrow{\tau}$ (see Section 2). Lemma 7 ensures that conditionally on $T_i = t$, $(\overrightarrow{\tau}^{(T_i-)}_x)_{x\geq 0}$ is independent of (G_i, R_i) and distributed as $(\overrightarrow{\tau}^{(t)}_x)_{x\geq 0}$. Then denoting by μ_t the law of (G_i, D_i) conditioned by $T_i = t$, we have

$$\mu_t(\mathrm{d}y\mathrm{d}x) = \mathbb{P}(G_t \in \mathrm{d}y,\ \overrightarrow{\tau}^{(t)}_{R_t} \in \mathrm{d}x), \tag{19}$$

where (G_t, R_t) is a random variable independent of $(\overrightarrow{\tau}^{(t)}_x)_{x\geq 0}$ and distributed as (G_i, R_i) conditioned on $T_i = t$.

By Lemma 6, $\big\{(T_i, G_i, D_i) : i \in \mathbb{N},\ T_i > t\big\}$ is independent of $\big\{(T_i, G_i, D_i) : i \in \mathbb{N}, T_i \leq t\big\}$. Then conditionally on $(T_i)_{i\in\mathbb{N}}$, $(G_i, D_i)_{i\in\mathbb{N}}$ are independent. Adding that $\{T_i : i \in \mathbb{N}\}$ is a PPP on $[0, 1/\mathrm{m}]$ with intensity $\mathrm{d}t\mathrm{m}/(1-\mathrm{m}t)$ ensures that $\big\{(T_i, G_i, D_i) : i \in \mathbb{N}\big\}$ is a (marked) PPP with intensity

$$\frac{\mathrm{m}}{1-\mathrm{m}t}\mathrm{d}t\mu_t(\mathrm{d}y\mathrm{d}x).$$

Furher, by (19), this intensity is eqaul to

$$\mathrm{d}t \int_0^\infty \mathbb{P}(\overrightarrow{\tau}^{(t)}_z \in \mathrm{d}x)\frac{\mathrm{m}}{1-\mathrm{m}t}\mathbb{P}(G_t \in \mathrm{d}y,\ R_t \in \mathrm{d}z) = \mathrm{d}t \int_0^\infty \mathbb{P}(\overrightarrow{\tau}^{(t)}_z \in \mathrm{d}x)\rho^{(t)}(\mathrm{d}y\mathrm{d}z)$$

using Theorem 3. This completes the proof. □

Proof of Theorem 4.
(i) Thanks to Lemma 6, $C^1(t+s) - C^1(t)$ is independent of $(g(u), d(u))_{u\in[0,t]}$.

(ii) We condition by $T_i \leq t \leq t+s < T_{i+1}$ for some $i \in \mathbb{N}$ and $l(t) = l$. Then $g(t+s) - g(t) = 0$ and no data arrived at the left of $\mathbf{B}_0(t)$ during the time interval $]t, t+s]$ is stored at the right of this block. So the increment $d(t+s) - d(t)$ is caused by files arriving on $\mathbf{B}_0(t)$: they are stored at the right on $\mathbf{B}_0(t)$ and may join data already stored. Note that we can change the order of arrival of files between t and $t+s$ (use identity (4)). Thus, we first store the files which arrive at the right of $d(t)$ between times t and $t+s$, then the files which arrive on $\mathbf{B}_0(t)$ between times t and $t+s$ and we forget the files which arrive at the left of $g(t)$.

STEP 1 : At time t, we consider the half hardware at the right of $d(t)$ which we identify with $[0,\infty[$. Its free space is equal to $\overrightarrow{\mathcal{R}(t)}$. We store the files $i \in \{i \in \mathbb{N} : t_i \in]t, t+s], x_i > d(t)\}$ on this half hardware $[0,\infty[$ at location $x_i - d(t)$ following the process described in Introduction (the size of the file i is still l_i). Following Section 2.1 in [1], we get the counterpart of the characterization of the free space (4). That is, the new free space of the half hardware is equal to $\{x \geq 0 : \widetilde{Y}_x = \widetilde{I}_x\}$, where for every $x \geq 0$,

$$\widetilde{Y}_x = -x + \sum_{\substack{0 \leq t_i \leq t+s \\ d(t) \leq x_i \leq d(t)+x}} l_i, \qquad \widetilde{I}_x := \inf\{\widetilde{Y}_y : 0 \leq y \leq x\}.$$

Using Lemma 1, we see that $\{(t_i, x_i - d(t), l_i) : x_i \geq d(t)\}$ is a PPP on $\mathbb{R}^{+3}$ with intensity $\mathrm{d}t \otimes \mathrm{d}x \otimes \nu(\mathrm{d}l)$. Then,

$$\big(\widetilde{Y}_x\big)_{x\geq 0} \stackrel{d}{=} \big(Y_x^{(t+s)}\big)_{x\geq 0}$$

is a Lévy process with Laplace exponent $\Psi^{(t+s)}$. As $[\Psi^{(t+s)}]'(0) < 0$, $\big(\widetilde{Y}_x\big)_{x\geq 0}$ is regular for $]-\infty, 0[$, in the sense that it takes negative values for some arbitrarily small x (Proposition 8 on page 84 in [2]). So for every stopping time T such that $\widetilde{Y}_T = \widetilde{I}_T$, there is the identity $T = \inf\{z \geq 0 : \widetilde{Y}_z < \widetilde{Y}_T\}$. This ensures that the free space $\{x \geq 0 : \widetilde{Y}_x = \widetilde{I}_x\}$ of the half hardware is the range of $(\widetilde{\tau}_x)_{x\geq 0}$ defined by

$$\widetilde{\tau}_x := \inf\{z \geq 0 : \widetilde{Y}_z < -x\}.$$

By Theorem 1 on page 189 in [2], $(\widetilde{\tau}_x)_{x\geq 0}$ is a subordinator with Laplace exponent $\kappa^{(t+s)}$, which is the inverse function of $-\Psi^{(t+s)}$. So $(\widetilde{\tau}_x)_{x\geq 0}$ is distributed as $(\overrightarrow{\tau}_x^{(t+s)})_{x\geq 0}$. By Lemma 1 again, $\{(t_i, x_i - d(t), l_i) : x_i > d(t)\}$ is independent of $(g(u), d(u))_{u\in[0,t]}$. So $(\widetilde{\tau}_x)_{x\geq 0}$ is independent of $(g(u), d(u))_{u\in[0,t]}$.

STEP 2 : To obtain the covering $\mathcal{C}(t+s)$, we now store the files $\{i : t_i \in]t, t+s], x_i \in [g(t), d(t)[\}$. It amounts to store these files in the first free spaces (i.e. as much on the left as possible) of the half hardware considered above, whose free space is the range of $(\widetilde{\tau}_x)_{x\geq 0}$. The variation of the right extremity is equal to the sum of the sizes of these files, say S_t^{t+s}, plus the sizes of the lengths of the blocks of the half hardware joined during their storage. That is, as for (18),

$$C^2(t+s) - C^2(t) = (0, \widetilde{\tau}_{S_t^{t+s}}), \qquad \text{where} \quad S_t^{t+s} := \sum_{\substack{t < t_i \leq t+s \\ x_i \in [g(t), d(t)[}} l_i.$$

Conditionally on $l(t) = l$, by Poissonian property, $S_t^{t+s} \stackrel{d}{=} S_{sl}$, where $(S_x)_{x\geq 0}$ is a subordinator with no drift and Lévy measure ν. Adding that S_t^{t+s} is independent of $(\widetilde{\tau}_x)_{x\geq 0}$ gives the law of $C^2(t+s) - C^2(t)$. As $(\widetilde{\tau}_x)_{x\geq 0}$ and S_t^{t+s} are independent of $(g(u), d(u))_{u\in[0,t]}$, so is $C^2(t+s) - C^2(t)$.

These properties ensure that $(g(t), d(t))_{t\in[0,1/\mathrm{m}[}$ is a Markov process. $\square$

To prove Corollary 3, we need the following result which uses notation of Theorem 4.

Lemma 8. *We have the following vague convergence of measure on* $]0,\infty[$ *:*

$$h^{-1}\mathbb{P}\big(\vec{\tau}^{(t)}_{S_{hl}} \in \mathrm{d}x\big) \stackrel{v}{\Longrightarrow} l\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\vec{\tau}^{(t)}_z \in \mathrm{d}x).$$

Proof. Denoting by ϕ the Laplace exponent of $(S_x)_{x\geq 0}$, $\big(\vec{\tau}^{(t)}_{S_{xl}}\big)_{x\geq 0}$ is a subordinator of Laplace exponent $l\phi\circ\kappa^{(t)}$ (see (2)). Moreover for every $\lambda \geq 0$, $\phi(\lambda) = \int_0^\infty (1-e^{-\lambda y})\nu(\mathrm{d}y)$, which entails that

$$\begin{aligned}
\phi\circ\kappa^{(t)}(\lambda) &= \int_0^\infty \big(1-e^{-z\kappa^{(t)}(\lambda)}\big)\nu(\mathrm{d}z) \\
&= \int_0^\infty \mathbb{E}\big(1-e^{-\lambda\vec{\tau}^{(t)}_z}\big)\nu(\mathrm{d}z) \\
&= \int_0^\infty (1-e^{-\lambda x})\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\vec{\tau}^{(t)}_z \in \mathrm{d}x).
\end{aligned}$$

Then $\big(\vec{\tau}^{(t)}_{S_{xl}}\big)_{x\geq 0}$ is a subordinator with no drift and Lévy measure

$$l\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\vec{\tau}^{(t)}_z \in \mathrm{d}x).$$

Using Exercise 1 Chapter I in [2] or [3] on page 8 completes the proof. $\square$

Proof of Corollary 3. We consider first the case when the increment of the left extremity is zero.

• Let $c > 0$ such that $\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\vec{\tau}^{(t)}_z = c) = 0$. Using Theorem 4 and recalling that $N_t^{t+h} = N_{]t+t+h]\times\mathbb{R}_+^2} = \mathrm{card}\{i \in \mathbb{N} : T_i \in]t,t+h]\}$, we have

$$P(g(t+h)-g(t) = 0,\ d(t+h)-d(t) \geq c \mid l(t) = l) = \mathbb{P}(N_t^{t+h} = 0)\mathbb{P}(\vec{\tau}^{(t)}_{S_{hl}} \geq c). \quad (20)$$

Adding that $\mathbb{P}(N_t^{t+h} = 0) \stackrel{h\to 0}{\longrightarrow} 1$ and using Lemma 8 give

$$h^{-1}P(g(t+h)-g(t) = 0,\ d(t+h)-d(t) \geq c \mid l(t) = l) \stackrel{h\to 0}{\longrightarrow} l\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\vec{\tau}^{(t)}_z \geq c). \quad (21)$$

• Let $a, b > 0$ and write

$$P(t,t+h) = \mathbb{P}(g(t)-g(t+h) \geq a,\ d(t+h)-d(t) \geq b \mid l(t) = l).$$

By Proposition 2, $\big\{(T_i, G_i, D_i) : i \in \mathbb{N}\big\}$ is a PPP on $[0, 1/\mathrm{m}] \times \mathbb{R}_+^2$ with intensity $\mathrm{d}t\mu^{(t)}(\mathrm{d}y\mathrm{d}x)$. The latter verifies $\mathbb{P}(N_t^{t+h} > 1) = o(h)$ $(h \to 0)$, so we have

$$h^{-1}\mathbb{P}(C^1(t+h)-C^1(t) \in]-\infty,-a]\times[b,\infty]) \stackrel{h\to 0}{\longrightarrow} \mu^{(t)}([a,\infty[\times[b,\infty]). \quad (22)$$

We can prove now that

$$\lim_{h\to 0} h^{-1}P(t,t+h) = \mu^{(t)}([a,\infty[\times[b,\infty[). \quad (23)$$

- First we give the lowerbound.

$$P(t,t+h) \geq \mathbb{P}(C^1(t+h)-C^1(t) \in]-\infty,-a] \times [b,\infty] \ \mid \ l(t)=l)$$

Using that $C^1(t+h)-C^1(t)$ is independent of $l(t)$ and (22), we get

$$\liminf_{h\to 0} h^{-1}P(t,t+h) \geq \mu^{(t)}([a,\infty[\times[b,\infty]). \tag{24}$$

- For the upperbound, observe that

$$\begin{aligned} P(t,t+h) \ &\leq \ \mathbb{P}(C^1(t+h)-C^1(t) \in]-\infty,-a] \times [b-\epsilon,\infty] \ \mid \ l(t)=l) \\ &+ \ \mathbb{P}(N_t^{t+h} \geq 1, \ C^2(t+h)-C^2(t) \in \{0\} \times [\epsilon,\infty[\ \mid \ l(t)=l). \end{aligned}$$

Using again $C^1(t+h)-C^1(t)$ is independent of $l(t)$ with (22) and Theorem 4 gives

$$\limsup_{h\to 0} h^{-1}P(t,t+h) \leq \mu^{(t)}([a,\infty[\times[b-\epsilon,\infty]).$$

Letting ϵ tend to 0 gives the upperbound :

$$\limsup_{h\to 0} h^{-1}P(t,t+h) \leq \mu^{(t)}([a,\infty[\times[b,\infty[).$$

The two limits (21) and (23) ensure the convergence of measures for sets of the form $\{0\}\times[c,d[$ (with $c>0$) and $[a,b[\times[c,d[$ (with $a>0$), which completes the proof. $\square$

7 Evolution of the right extremity and of the length

Proposition 2, Theorem 4 and Corollary 3 give by projection :

Corollary 4. $(d(t))_{t\in[0,1/\mathrm{m}[}$ *is a jump process satisfying*

(i) $\{(T_i,D_i) : i\in\mathbb{N}\}$ *is a PPP on* $[0,1/\mathrm{m}[\times\mathbb{R}^+$ *with intensity*

$$\frac{\mathrm{d}t \int_{z\in[0,\infty]} \mathrm{d}z\bar{\nu}(z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}x)}{1-\mathrm{m}t},$$

and $\{(T_i,D_i) : i\in\mathbb{N}, \ T_i>t\}$ *is independent of* $(d(u))_{u\in[0,t]}$.

(ii) For all $0\leq t\leq t+s<1/\mathrm{m}$ *:*
Conditionally on $l(t)=l$, $d(t+s)-d(t)$ *is independent of* $(d(u))_{u\in[0,t]}$.
Conditionally also on $T_i\leq t\leq t+s<T_{i+1}$ *for some* $i\in\mathbb{N}$ *:*

$$d(t+s)-d(t) \stackrel{d}{=} \overrightarrow{\tau}_{S_{sl}}^{(t+s)},$$

where $(S_x)_{x\geq 0}$ *is a subordinator with no drift and Lévy measure* ν*, that is independent of* $(\overrightarrow{\tau}_x^{(t+s)})_{x\geq 0}$.

The jump rate of $(d(t))_{t\in[0,1/\mathrm{m}[}$ *is given by the following vague convergence of measures on* $]0,\infty[$ *for* h *tending to* 0 *:*

$$\frac{\mathbb{P}(d(t+h)-d(t)\in\mathrm{d}x \ \mid \ l(t)=l)}{h} \xrightarrow{w} \frac{\int_0^\infty \mathrm{d}z\bar{\nu}(z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)}\in\mathrm{d}x)}{1-\mathrm{m}t} + l\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)}\in\mathrm{d}x).$$

We stress that $(d(t))_{t\in[0,1/\mathrm{m}[}$ is not a Markov process since the jumps D_i before time t give informations about $l(t)$ and thus about the future of the process.
Note also that we can derive the law of $d(t)$ conditionally on $l(t)$ using Theorem 1. More precisely, conditionally on $l(t)=l$,

$$\forall d>0, \qquad \mathbb{P}(l(t)\in \mathrm{d}l \ \mid \ d(t)=d)=\mathbf{1}_{l\geq d}\frac{\Pi^{(t)}(\mathrm{d}l)}{\bar{\Pi}^{(t)}(d)}.$$

Finally we turn our interest to the process of the length $(l(t))_{t\in[0,1/\mathrm{m}]}$. Its increments which are due to files arrived at the left of $g(t)$ which are not stored entirely at the left $g(t)$, are denoted by L_i :

$$L_i:=l(T_i)-l(T_i^-)=G_i+D_i.$$

The other increments of $(l(t))_{t\in[0,1/\mathrm{m}]}$ are due to files which arrive on $\mathbf{B}_0$. We can view $(l(t))_{t\in[0,1/\mathrm{m}]}$ as a branching process in continuous time with immigration L_i at time T_i (with no death, inhomogeneous branching and inhomogeneous immigration) :

Corollary 5. *$(l(t))_{t\in[0,1/\mathrm{m}[}$ is an inhomogeneous pure jump Markov process satisfying*

(i) $\{(T_i,L_i): i\in\mathbb{N}\}$ is a PPP on $[0,1/\mathrm{m}[\times\mathbb{R}^+$ with intensity

$$\mathrm{d}t\int_{z\in[0,\infty]}\nu(\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)}\in \mathrm{d}x)x,$$

and $\{(T_i,L_i): i\in\mathbb{N}, T_i>t\}$ is independent of $(l(s))_{s\in[0,t]}$

(ii) Conditionally on $T_i\leq t\leq t+s<T_{i+1}$ for some $i\in\mathbb{N}$, $(l(t+u))_{u\in[0,t-s]}$ satisfies the branching property : the law of $(l(t+u))_{u\in[0,t-s]}$ conditioned on $l(t)=x+y$ is equal to the law of the sum of two independent processes whose laws are respectively equal to $(l(t+u))_{u\in[0,t-s]}$ conditioned on $l(t)=x$ and $(l(t+u))_{u\in[0,t-s]}$ conditioned on $l(t)=y$.

The jump rate of $(l(t))_{t\in[0,1/\mathrm{m}[}$ is given by the following vague convergence of measures on $]0,\infty[$ for h tending to 0 :

$$\frac{\mathbb{P}(l(t+h)-l(t)\in \mathrm{d}x \ \mid \ l(t)=l)}{h} \ \overset{w}{\Longrightarrow} \ (x+l)\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)}\in \mathrm{d}x).$$

Example 3. For the basic example $\nu=\delta_1$, the jump rate of the lenght is equal to

$$\sum_{n=1}^{\infty}\frac{n+l}{n}e^{-tn}\frac{(tn)^{n-1}}{(n-1)!}\delta_n(\mathrm{d}x).$$

This is a consequence of the last displayed limit and (8).

Proof of Corollary 4. Using (17), we get :

$$\int_{z\in[0,\infty]} \mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}x) \int_{y\in[0,\infty]} \rho^{(t)}(\mathrm{d}y\mathrm{d}z) = \frac{\int_{z\in[0,\infty]} \mathrm{d}z\bar{\nu}(z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}x)}{1-\mathrm{m}t},$$

which gives the intensity of $\{(T_i, D_i) : i \in \mathbb{N}\}$ by Proposition 2. $\square$

Proof of Corollary 5. (i) Writing $L_i = G_i + D_i$, Proposition 2 entails that $\{(T_i, L_i) : i \in \mathbb{N}\}$ is a PPP on $[0, 1/\mathrm{m}] \times \mathbb{R}^+$ with intensity $\mathrm{d}t\widetilde{\mu}_t(\mathrm{d}x)$ where $\widetilde{\mu}_t$ is a measure on $\mathbb{R}^+$ defined for a Borel set A of $\mathbb{R}^+$ by

$$\widetilde{\mu}_t(A) = \int_{\mathbb{R}_+^2} \mathbb{1}_{\{y+y'\in A\}} \int_0^\infty \mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}y')\rho^{(t)}(\mathrm{d}y\mathrm{d}z).$$

To determine $\widetilde{\mu}_t$, we compute its Laplace transform using Lemma 3 :

$$\begin{aligned}
\int_0^\infty e^{-\lambda x}\widetilde{\mu}_t(\mathrm{d}x) &= \int_{\mathbb{R}^{+3}} e^{-\lambda(y+y')}\rho^{(t)}(\mathrm{d}y\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}y') \\
&= \int_{\mathbb{R}^{+3}} e^{-\lambda y'}\mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}y')\mathrm{d}z \int_z^\infty \nu(\mathrm{d}l)\big[e^{-\lambda y}\mathbb{P}(\overrightarrow{\tau}_{l-z}^{(t)} \in \mathrm{d}y) \\
&\qquad + \int_0^y e^{-\lambda x}\mathbb{P}(\overrightarrow{\tau}_{l-z}^{(t)} \in \mathrm{d}x)(y-x)e^{-\lambda(y-x)}\Pi^{(t)}(\mathrm{d}y-x)\big] \\
&= \int_0^\infty \mathrm{d}z e^{-z\kappa^{(t)}(\lambda)} \int_z^\infty \nu(\mathrm{d}l)e^{-(l-z)\kappa^{(t)}(\lambda)}\big[1 + \int_0^\infty e^{-\lambda u}u\Pi^{(t)}(\mathrm{d}u)\big] \\
&= \int_0^\infty \nu(\mathrm{d}l)le^{-l\kappa^{(t)}(\lambda)}[\kappa^{(t)}]'(\lambda) \\
&= -\frac{\partial}{\partial y}\left[\int_0^\infty \nu(\mathrm{d}l)e^{-l\kappa^{(t)}(y)}\right](\lambda) \\
&= -\frac{\partial}{\partial y}\left[\int_0^\infty e^{-yx}\int_0^\infty \nu(\mathrm{d}l)\mathbb{P}(\overrightarrow{\tau}_l^{(t)} \in \mathrm{d}x)\right](\lambda) \\
&= \int_0^\infty e^{-\lambda x}x\int_0^\infty \nu(\mathrm{d}l)\mathbb{P}(\overrightarrow{\tau}_l^{(t)} \in \mathrm{d}x).
\end{aligned}$$

Then $\widetilde{\mu}_t(\mathrm{d}x) = x\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}_z^{(t)} \in \mathrm{d}x)$, which gives the intensity of $\{(T_i, L_i) : i \in \mathbb{N}\}$.

(ii) The branching property can be seen as a consequence of the determination of the jump rate. We give here a more intuitive approach : We condition by $l(t) = x+y$ and by $T_i \leq t \leq t+s < T_{i+1}$ and we make the decomposition effective by splitting $\mathbf{B}_0(t)$ in two segments of length x and y. First we store the files $\{i : t_i \in]t, t+s], x_i > d(t)\}$. The free space of the half line at the right of $\mathbf{B}_0(t)$ is now the closed range a subordinator distributed like $(\overrightarrow{\tau}_x^{(t+s)})_{x\geq 0}$ (see STEP1 in the proof of Corollary 4). Then we store successively the files $\{i : t_i \in]t, t+s], x_i \in [g(t), g(t)+x]\}$ and $\{i : t_i \in]t, t+s], x_i \in]g(t)+x, d(t)]\}$ which induce two successive increments of the length. The free space at the right of 0 after the first storage keeps the same distribution and is independent of the first increment by strong regeneration. So the two increments are independent and distributed respectively like $l(t+s)-l(t)$ conditioned by $l(t)=x$ and by $l(t)=y$. This gives the result since $l(t)$ is Markovian. Formally $l(t+s)-l(t)$ is equal to $\overrightarrow{\tau}^{(t+s)}_{S_{s(x+y)}}$ (see proof of Proposition 2) and

$$\overrightarrow{\tau}^{(t+s)}_{S_{s(x+y)}} = \overrightarrow{\tau}^{(t+s)}_{S_{sx}} + \overrightarrow{\tau}^{(t+s)}_{S_{s(x+y)}} - \overrightarrow{\tau}^{(t+s)}_{S_{sx}}$$

gives the decomposition expected since $\overrightarrow{\tau}^{(t+s)}_{S_{s(x+y)}} - \overrightarrow{\tau}^{(t+s)}_{S_{sx}} \stackrel{d}{=} \overrightarrow{\tau}^{(t+s)}_{S_{sy}}$.

Using Corollary 3 and recalling the definition of $\widetilde{\mu}_t$ given at the beginning of the proof ensures that $h^{-1}\mathbb{P}(l(t+h)-l(t)\in \mathrm{d}x \mid l(t)=l)$ converges to

$$\widetilde{\mu}_t(\mathrm{d}x) + l\int_0^\infty \nu(\mathrm{d}z)\mathbb{P}(\overrightarrow{\tau}^{(t)}_z \in \mathrm{d}x).$$

The completes the proof, since $\widetilde{\mu}$ has been determined above. □

8 Complements

8.1 Distribution of $\{(T_i, G_i) : i \in \mathbb{N}\}$ derived from Theorem 3

In Section 5, we used the total intensity of the PPP $\{(T_i, G_i) : i \in \mathbb{N}\}$ to prove that the intensity of the PPP $\{(T_i, G_i, R_i) : i \in \mathbb{N}\}$ is equal to $\mathrm{d}t\rho^{(t)}(\mathrm{d}y\mathrm{d}z)$ (Theorem 3). Here we check that integrating this intensity with respect to the third coordinate enables us to recover the intensity of $\{(T_i, G_i) : i \in \mathbb{N}\}$ given in Theorem 2.
For that purpose, use Lemma 3 to rewrite $\rho^{(t)}$ as

$$\rho^{(t)}(\mathrm{d}y\mathrm{d}z) = \mathrm{d}z\int_0^\infty \nu(\mathrm{d}l+z)\big(\mathbb{P}(\overleftarrow{\tau}^{(t)}_l \in \mathrm{d}y) + \int_0^y \mathbb{P}(\overleftarrow{\tau}^{(t)}_l \in \mathrm{d}x)(y-x)\Pi^{(t)}(\mathrm{d}y-x)\big)$$

and calculate the Laplace transform of $\int_{z\in[0,\infty]}\rho^{(t)}(\mathrm{d}y\mathrm{d}z)$.

$$\begin{aligned}
&\int_{y\in[0,\infty]} e^{-\lambda y}\int_{z\in[0,\infty]} \rho^{(t)}(\mathrm{d}y\mathrm{d}z)\\
=\ &\int_0^\infty\int_0^\infty \mathrm{d}z\nu(\mathrm{d}l+z)\int_0^\infty e^{-\lambda y}\big[\mathbb{P}(\overleftarrow{\tau}^{(t)}_l \in dy) + \int_0^y \mathbb{P}(\overleftarrow{\tau}^{(t)}_l \in \mathrm{d}x)(y-x)\Pi^{(t)}(\mathrm{d}y-x)\big]\\
=\ &\int_0^\infty \mathrm{d}l\bar{\nu}(l)\big[e^{-l\kappa(\lambda)} + \int_0^\infty \mathbb{P}(\overleftarrow{\tau}^{(t)}_l \in \mathrm{d}x)e^{-\lambda x}\int_x^\infty e^{-\lambda(y-x)}(y-x)\Pi^{(t)}(\mathrm{d}y-x)\big]\\
=\ &\int_0^\infty \mathrm{d}l\bar{\nu}(l)e^{-l\kappa(\lambda)}[\kappa^{(t)}]'(\lambda)\\
=\ &\int_0^\infty \mathrm{d}l\frac{\bar{\nu}(l)}{l}\frac{\partial}{\partial\lambda}\mathbb{E}(-e^{-l\kappa^{(t)}(\lambda)})\\
=\ &\int_0^\infty \mathrm{d}l\frac{\bar{\nu}(l)}{l}\frac{\partial}{\partial\lambda}\mathbb{E}(-e^{-\lambda\overleftarrow{\tau}^{(t)}_l})\\
=\ &\int_0^\infty \mathrm{d}l\frac{\bar{\nu}(l)}{l}\int_0^\infty e^{-\lambda y}y\mathbb{P}(\overleftarrow{\tau}^{(t)}_l \in \mathrm{d}y)\\
=\ &\int_0^\infty \mathrm{d}ye^{-\lambda y}\int_0^\infty \mathbb{P}(Y^{(t)}_y \in -\mathrm{d}l)\bar{\nu}(l) \quad \text{using (10).}
\end{aligned}$$

Thus, we conclude with

$$\mathrm{d}t \int_{z\in[0,\infty]} \rho^{(t)}(\mathrm{d}y\mathrm{d}z) = \mathrm{d}t\mathrm{d}x \int_0^\infty \mathbb{P}(Y_x^{(t)} \in -\mathrm{d}l)\bar{\nu}(l).$$

8.2 Direct proof of Corollary 2 using fluctuation theory

Here we determine the distribution of the remaining data using fluctuation theory : we get laws at fixed times and do not need Theorem 2, as for the proof of Section 5.

We fix t,h and $x \geq 0$. We add the lengths of files fallen in $[g(t) - x, g(t)]$ during the time interval $]t, t+h]$. Then we remove the free space in $[g(t) - x, g(t)]$ at time t which is equal to $L_x^{(t)}$. The sum of data arrived at the left of $\mathbf{B}_0(t)$ not stored at the left of $\mathbf{B}_0(t)$ between time t and $t+h$ is equal to the maximum in $x \geq 0$ of this difference. It is also the quantity of data which has tried to occupy the location $g(t)$ (successfully or not) between time t and $t+h$: $Y_{g(t)}^{(t+h)} - I_{g(t)}^{(t+h)}$. So, we have

Lemma 9. *Let* $0 \leq t < 1/\mathrm{m}$ *and* $h \geq 0$*, then*

$$Y_{g(t)}^{(t+h)} - I_{g(t)}^{(t+h)} = \sup\{S_{hx} - L_x^{(t)}, x \geq 0\} = \sup\{S_{h\overleftarrow{\tau}_x^{(t)}} - x, x \geq 0\} \qquad a.s,$$

where $(S_x)_{x\geq 0}$ *is a subordinator with drift* $\mathrm{d} = 0$ *and Lévy measure* $\nu(dx)$*, which is independent of* $(L_x^{(t)})_{x\geq 0}$ *and* $(\overleftarrow{\tau}_x^{(t)})_{x\geq 0}$*.*

Denoting $S^{(t,h)} := \sup\{S_{h\overleftarrow{\tau}_x^{(t)}} - x, x \geq 0\}$, we have for all $0 < a \leq b$,

$$\lim_{h\to 0} h^{-1}\mathbb{P}(S^{(t,h)} \in [a,b]) = \lim_{h\to 0} h^{-1}\mathbb{P}(\exists i \in \mathbb{N} : (T_i, R_i) \in]t, t+h] \times [a,b])$$

and we find the law given in Corollary 2 :

Proposition 3. *We have the following weak convergence of bounded measures on* $]0,\infty[$ *when* h *tends to* 0 *:*

$$\frac{\mathbb{P}(S^{(t,h)} \in \mathrm{d}x)}{h} \overset{w}{\Longrightarrow} \frac{\bar{\nu}(x)\mathrm{d}x}{1 - \mathrm{m}t}.$$

Proof. $(S_{h\overleftarrow{\tau}_x^{(t)}} - x)_{x\geq 0}$ is a lévy process with negative drift -1, no negative jumps and bounded variation. Its Laplace exponent is $\kappa^{(t)} \circ (h\phi) - id$, where ϕ is the Laplace exponent of S and is defined by

$$\forall \lambda \geq 0, \quad \phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\nu(\mathrm{d}x).$$

Note also that using (9), we have

$$\big[\kappa^{(t)} \circ (h\phi) - id\big]'(0) = [\kappa^{(t)}]'(0).h.\phi'(0) - 1 = \frac{1}{1-\mathrm{m}t}\mathrm{m}h - 1, \tag{25}$$

which is negative since $0 \le t+h < 1/\mathrm{m}$. Then identity (14) in [1] or Theorem 5 in [2] ensure that $\forall \lambda > 0,\ \forall h \in [0, 1/\mathrm{m} - t[$,

$$\mathbb{E}\big(\exp(-\lambda S^{(t,h)})\big) = \left(\frac{1}{1-\mathrm{m}t}\mathrm{m}h - 1\right)\frac{\lambda}{(\kappa^{(t)} \circ (h\phi) - id)(\lambda)}$$

Moreover,

$$\frac{(\kappa^{(t)} \circ (h\phi) - id)(\lambda)}{\lambda} = \frac{\kappa^{(t)}(h\phi(\lambda))}{h\phi(\lambda)}\frac{h\phi(\lambda)}{\lambda} - 1 = -1 + \frac{1}{1-\mathrm{m}t}\frac{h\phi(\lambda)}{\lambda} + \circ_{h\to 0}(h).$$

So

$$\mathbb{E}\big(\exp(-\lambda S^{(t,h)})\big) = 1 + \frac{1}{1-\mathrm{m}t}\big(\frac{\phi(\lambda)}{\lambda} - \mathrm{m}\big)h + \circ_{h\to 0}(h).$$

We can now prove the convergence of $h^{-1}\mathbb{P}(S^{(t,h)} > x)$ when h tends to 0.

$$\lim_{h\to 0}\int_0^\infty e^{-\lambda x}\frac{\mathbb{P}(S^{(t,h)} > x)}{h}\mathrm{d}x = \lim_{h\to 0}\frac{1-\mathbb{E}\big(\exp(-\lambda S^{(t,h)})\big)}{h\lambda} = \frac{1}{1-\mathrm{m}t}\big(\frac{\mathrm{m}}{\lambda} - \frac{\phi(\lambda)}{\lambda^2}\big).$$

Moreover Fubini gives

$$\int_0^\infty \mathrm{d}x e^{-\lambda x}\int_x^\infty \bar{\nu}(a)\mathrm{d}a = \int_0^\infty \nu(\mathrm{d}y)\int_0^y \mathrm{d}a\frac{1-e^{-\lambda a}}{\lambda} = \frac{\mathrm{m}}{\lambda} - \frac{\phi(\lambda)}{\lambda^2}.$$

Then for every $\lambda > 0$,

$$\lim_{h\to 0}\int_0^\infty e^{-\lambda x}\frac{\mathbb{P}(S^{(t,h)} > x)}{h}\mathrm{d}x = \int_0^\infty e^{-\lambda x}\frac{\int_x^\infty \bar{\nu}(a)\mathrm{d}a}{1-\mathrm{m}t}\mathrm{d}x,$$

which proves the convergence of $\mathbb{P}(S^{(t,h)} \in \mathrm{d}x)/h$ to $\bar{\nu}(x)\mathrm{d}x/(1-\mathrm{m}t)$. Indeed, introduce the measures $\mu_h(\mathrm{d}x)$ and $\mu(\mathrm{d}x)$ on $\mathbb{R}^+$ whose tails are given by

$$\mu_h(]x,\infty]) = e^{-x}\mathbb{P}(S^{(t,h)} > x)/h, \qquad \mu(]x,\infty]) = e^{-x}\int_x^\infty \bar{\nu}(a)\mathrm{d}a/(1-\mathrm{m}t).$$

The last displayed limit entails the weak convergence of $\mu_h(\mathrm{d}x)$ to $\mu(\mathrm{d}x)$ when h tends to 0, by convergence of Laplace transforms. As μ is non atomic, for every $x \ge 0$, $\mu_h(]x,\infty])$ tends to $\mu(]x,\infty])$, which proves that $\mathbb{P}(S^{(t,h)} > x)/h$ tends to $\int_x^\infty \bar{\nu}(a)\mathrm{d}a/(1-\mathrm{m}t)$. $\square$

Remark 3. Denote $\gamma^{(t,h)}$ the a.s instant at which the supremum $S^{(t,h)}$ is reached. To obtain the distribution of $\{(T_i, G_i, R_i) \ :\ i \in \mathbb{N}\}$ by this way, we need to know the joint law of $(S^{(t,h)}, \overleftarrow{\tau}^{(t)}_{\gamma^{(t,h)}})$ which we cannot derive directly from fluctuation theory.

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